The N-Stable Category

Jeremy Rollin Bundick Brightbill

UCSB

January 5, 2022

Introduction

This talk is based on a joint work with Vanessa Miemietz (UEA)

Outline

Exact Categories

Reference

- Bühler (2010). *Exact categories*.

Exact Categories

Let $\mathcal{E}$ be an additive category.

- A **kernel-cokernel pair** in $\mathcal{E}$ is a diagram $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ such that $f = ker(g)$ and $g = coker(f)$.
- An **exact category** $(\mathcal{E}, \mathcal{S})$ is the data of an additive category $\mathcal{E}$ and a collection $\mathcal{S}$ of kernel-cokernel pairs in $\mathcal{E}$ (called the **admissible exact sequences**) satisfying certain axioms.

Frobenius Exact Categories

Let $\mathcal{E}$ an exact category.

- $\mathcal{E}$ is **Frobenius** if projective and injective objects coincide, and there are enough of each.
- In this case, define the **stable category of** $\mathcal{E}$, $\mathtt{stab}(\mathcal{E})$, to be the additive quotient $\mathcal{E} \,/\, \mathsf{Proj}(\mathcal{E})$.
- Objects: Same as $\mathcal{E}$
- Morphisms: $\mathsf{Hom}_{\mathcal{E}}(X, Y)/\mathcal{P}(X, Y)$, where $\mathcal{P}(X, Y)$ is the subgroup of morphisms factoring through a projective-injective object.
- $\mathtt{stab}(\mathcal{E})$ is a triangulated category with suspension functor Ω^{-1} defined by

$$X \rightarrowtail I_X \twoheadrightarrow \Omega^{-1} X$$

N-Complexes

Reference

- Iyama, Kato, Miyachi (2017). *Derived categories of N-complexes*.

Complexes

Let $\mathcal{A}$ denote an additive category.

- Category of complexes over $\mathcal{A}$: $C(\mathcal{A})$
- Objects: $(X^\bullet, d_X^\bullet)$

$$\cdots \xrightarrow{d_X^{-2}} X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots$$

- Differential satisfies $d^2 = 0$
- Morphisms: $f^\bullet : X^\bullet \to Y^\bullet$

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_X^{-2}} & X^{-1} & \xrightarrow{d_X^{-1}} & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots \\ & & \downarrow f^{-1} & & \downarrow f^0 & & \downarrow f^1 & & \\ \cdots & \xrightarrow{d_Y^{-2}} & Y^{-1} & \xrightarrow{d_Y^{-1}} & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots \end{array}$$

N-Complexes

Let $\mathcal{A}$ denote an additive category.

- Category of N-complexes over $\mathcal{A}$: $C_N(\mathcal{A})$
- Objects: $(X^\bullet, d_X^\bullet)$

$$\cdots \xrightarrow{d_X^{-2}} X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots$$

- Differential satisfies $d_X^N = 0$
- Morphisms: $f^\bullet : X^\bullet \to Y^\bullet$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_X^{-2}} & X^{-1} & \xrightarrow{d_X^{-1}} & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots \\
 & & \downarrow f^{-1} & & \downarrow f^0 & & \downarrow f^1 & & \\
\cdots & \xrightarrow{d_Y^{-2}} & Y^{-1} & \xrightarrow{d_Y^{-1}} & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots
\end{array}$$

Homotopy Category of Complexes

- We say the sequence $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet$ is **chainwise split exact** if $X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n$ is split exact for each n.
- The chainwise split exact sequences give $C(\mathcal{A})$ the structure of a Frobenius exact category.
- Projective-injective objects are direct sums of shifts of complexes of the form

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{id_X} & X & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & 0 & & 1 & & & & \end{array}$$

- The stable category of $C(\mathcal{A})$ is the **homotopy category of complexes**, $K(\mathcal{A})$

Homotopy Category of N-Complexes

- We say the sequence $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet$ is **chainwise split exact** if $X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n$ is split exact for each n.
- The chainwise split exact sequences give $C_N(\mathcal{A})$ the structure of a Frobenius exact category.
- Projective-injective objects are direct sums of shifts of complexes of the form

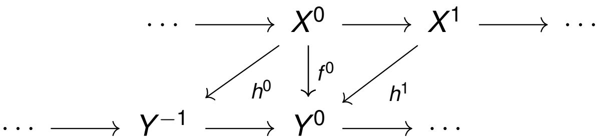
$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{id_X} & \cdots & \xrightarrow{id_X} & X & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & 0 & & \cdots & & N-1 & & & & \end{array}$$

- The stable category of $C_N(\mathcal{A})$ is the **homotopy category of N-complexes**, $K_N(\mathcal{A})$

Null-homotopic maps

- $f^\bullet : X^\bullet \to Y^\bullet$ is **null-homotopic** if it factors through a projective-injective object
- Equivalently, there exist morphisms $h^i : X^i \to Y^{i-1}$ such that

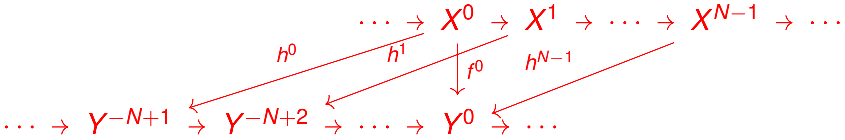
$$f^k = d_Y^{k-1} \circ h^k + h^{k+1} \circ d_X^k$$



N-Null-homotopic maps

- $f^\bullet : X^\bullet \to Y^\bullet$ is **null-homotopic** if it factors through a projective-injective object
- Equivalently, there exist morphisms $h^k : X^k \to Y^{k-(N-1)}$ such that

$$f^k = \sum_{j=1}^{N} d_Y^{k+j-N,N-j} \circ h^{k+j-1} \circ d_X^{k,j-1}$$



Suspension Functor in $K(\mathcal{A})$

- The shift functor [1] can be computed via the chainwise split exact sequence $C^\bullet \rightarrowtail I^\bullet \twoheadrightarrow C^\bullet[1]$

$$
\begin{array}{ccccccccccccc}
C^\bullet & = & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
I^\bullet & = & \cdots & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{id_X} & X & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C^\bullet[1] & = & \cdots & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & & & -1 & & 0 & & & &
\end{array}
$$

Suspension Functor in $K_N(\mathcal{A})$

- The suspension functor Σ can be computed via the chainwise split exact sequence $C^\bullet \rightarrowtail I_N^\bullet \twoheadrightarrow \Sigma C^\bullet$

$$\begin{array}{ccccccccccccc}
C^\bullet & = \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
I_N^\bullet & = \cdots \longrightarrow & 0 & \longrightarrow & X & \longrightarrow \cdots \longrightarrow & X & \xrightarrow{id_X} & X & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Sigma C^\bullet & = \cdots \longrightarrow & 0 & \longrightarrow & X & \longrightarrow \cdots \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & -N+1 & & -1 & & 0 & & & &
\end{array}$$

Suspension Functor, Cont'd

- $\Sigma \ncong [1]$, but $\Sigma^2 \cong [N]$ in $K_N(\mathcal{A})$.

$$
\begin{array}{ccccccccccccccc}
\Sigma C^\bullet & = \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow \cdots \longrightarrow & X & \longrightarrow & 0 & \longrightarrow \cdots \\
\rightarrowtail\!\!\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
I_N^\bullet[1] & = \cdots \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & X & \longrightarrow \cdots \longrightarrow & X & \longrightarrow & 0 & \longrightarrow \cdots \\
\twoheadrightarrow\!\!\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\Sigma^2 C^\bullet & = \cdots \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
 & & & & -N & & -N+1 & & -1 & & 0 &
\end{array}
$$

Homology

Let $\mathcal{A}$ be an abelian category.

- Let $(X^\bullet, d_X^\bullet) \in C(\mathcal{A})$.
- Cycles: $Z^n(X^\bullet) = ker(d_X^n)$
- Boundaries: $B^n(X^\bullet) = im(d_X^{n-1})$
- Homology: $H^n(X^\bullet) = Z^n(X^\bullet)/B^n(X^\bullet)$

N-Homology

Let $\mathcal{A}$ be an abelian category.

- Let $(X^\bullet, d_X^\bullet) \in C_N(\mathcal{A})$.
- Cycles: $Z_r^n(X^\bullet) = ker(d_X^{n,r})$
- Boundaries: $B_r^n(X^\bullet) = im(d_X^{n-(N-r),N-r})$
- Homology: $H_r^n(X^\bullet) = Z_r^n(X^\bullet)/B_r^n(X^\bullet)$
- $d_X^{n,r} := d_X^{n+r-1} \cdots d_X^n$ is the composition of r successive differentials, starting at d_X^n.

The Derived Category of Complexes

- $X^\bullet \in K(\mathcal{A})$ is **acyclic** if $H^n(X^\bullet) = 0$ for all $n \in \mathbb{Z}$.
- Acyclic complexes form a thick subcategory of $K^{ac}(\mathcal{A}) \subseteq K(\mathcal{A})$.
- The derived category is the Verdier quotient $D(A) = K(\mathcal{A})/K^{ac}(\mathcal{A})$.

The Derived Category of N-Complexes

- $X^\bullet \in H_N(\mathcal{A})$ is **acyclic** if $H_r^n(X^\bullet) = 0$ for all $n \in \mathbb{Z}$, $1 \leq r \leq N-1$.
- Acyclic complexes form a thick subcategory $K_N^{ac}(\mathcal{A}) \subseteq K_N(\mathcal{A})$.
- The derived category is the Verdier quotient $D_N(A) = K_N(\mathcal{A})/K_N^{ac}(\mathcal{A})$.

Buchweitz's Theorem

Buchweitz's Theorem

Let $\mathcal{A}$ be an abelian category which is Frobenius exact.

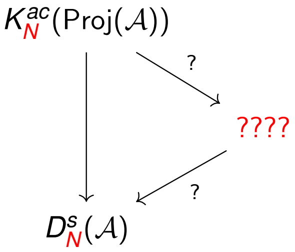
- The perfect derived category, $D^{perf}(\mathcal{A})$ is the image of $K^b(\text{Proj}(\mathcal{A}))$ in $D^b(\mathcal{A})$.
- The singularity category $D^s(\mathcal{A})$ is the Verdier quotient $D^b(\mathcal{A})/D^{perf}(\mathcal{A})$.
- There are equivalences of categories

$$\begin{array}{ccc} K^{ac}(\text{Proj}(\mathcal{A})) & \xrightarrow{Z^0} & \text{stab}(\mathcal{A}) \\ \downarrow & \swarrow{\scriptstyle \iota} & \\ D^s(\mathcal{A}) & & \end{array}$$

Buchweitz's Theorem for N-Complexes

Let $\mathcal{A}$ be an abelian category which is Frobenius exact.

- The perfect derived category, $D_N^{perf}(\mathcal{A})$ is the image of $K_N^b(\mathrm{Proj}(\mathcal{A}))$ in $D_N^b(\mathcal{A})$.
- The singularity category $D_N^s(\mathcal{A})$ is the Verdier quotient $D_N^b(\mathcal{A})/D_N^{perf}(\mathcal{A})$.
- There are equivalences of categories



The N-Stable Category

The Monomorphism Category

Let $\mathcal{E}$ be an exact category. Let $k \geq 0$.

- Let $\mathrm{MMor}_k(\mathcal{E})$ be the category whose objects are diagrams

$$(X_\bullet, \alpha_\bullet) = X_1 \overset{\alpha_1}{\rightarrowtail} X_2 \overset{\alpha_2}{\rightarrowtail} \cdots \overset{\alpha_k}{\rightarrowtail} X_{k+1}$$

  of k successive admissible monomorphisms.

- A morphism $f_\bullet : (X_\bullet, \alpha_\bullet) \to (Y_\bullet, \beta_\bullet)$ is a collection of morphisms $f_i : X_i \to Y_i$ making the obvious diagram commute:

$$\begin{array}{ccccccc} X_1 & \overset{\alpha_1}{\rightarrowtail} & X_2 & \overset{\alpha_2}{\rightarrowtail} & \cdots & \overset{\alpha_k}{\rightarrowtail} & X_{k+1} \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_{k+1}} \\ Y_1 & \overset{\beta_1}{\rightarrowtail} & Y_2 & \overset{\beta_2}{\rightarrowtail} & \cdots & \overset{\beta_k}{\rightarrowtail} & Y_{k+1} \end{array}$$

- Define the (epi)morphism category $\mathrm{(E)Mor}_k(\mathcal{E})$ similarly.

The Exact Structure on the Monomorphism Category

- Define an admissible short exact sequence in $\mathrm{MMor}_k(\mathcal{E})$ to be a kernel-cokernel pair $X_\bullet \overset{f_\bullet}{\rightarrowtail} Y_\bullet \overset{g_\bullet}{\twoheadrightarrow} Z_\bullet$ such that each f_i is an admissible monomorphism and each g_i is an admissible epimorphism in $\mathcal{E}$.

Theorem (B., Miemietz, 2021)

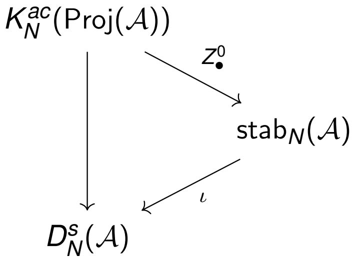
Let $\mathcal{E}$ be an exact category and let $k \geq 0$. Then $\mathrm{MMor}_k(\mathcal{E})$ is an exact category. If $\mathcal{E}$ is Frobenius, so is $\mathrm{MMor}_k(\mathcal{E})$.

- Define the ***N*-stable category of** $\mathcal{E}$, $\mathrm{stab}_N(\mathcal{E})$, to be the stable category of $\mathrm{MMor}_{N-2}(\mathcal{E})$.
- $(X_\bullet, \alpha_\bullet) \in \mathrm{MMor}_k(\mathcal{A})$ is projective (injective) iff each X_i is projective (injective) and each α_i is split.

Buchweitz's Theorem for N-Complexes

Theorem (B., Miemietz, 2021)

Let $\mathcal{A}$ be an abelian category which is Frobenius exact. There are equivalences of categories



- $Z_\bullet^0(X^\bullet) = Z_1^0(X^\bullet) \hookrightarrow Z_2^0(X^\bullet) \hookrightarrow \cdots \hookrightarrow Z_{N-1}^0(X^\bullet)$
- ι includes $\mathrm{stab}_N(\mathcal{A})$ as N-complexes concentrated in degrees 1 through $N-1$.

Fractional Calabi-Yau Categories

Related Work

- Ringel, Schmidmeier (2008). *The Auslander-Reiten translation in submodule categories*.
- Xiong, Zhang, and Zhang (2014). *Auslander-Reiten translations in monomorphism categories*.

Rotation Functor

Let A be a finite-dimensional, self-injective k-algebra.

- Define the cokernel functor $\mathsf{Cok} : \mathsf{MMor}_{N-2}(A) \to \mathsf{EMor}_{N-2}(A)$ be given by

$$\mathsf{Cok}(X_1 \hookrightarrow \cdots \hookrightarrow X_{N-1}) = X_{N-1} \twoheadrightarrow X_{N-1}/X_1 \twoheadrightarrow \cdots \twoheadrightarrow X_{N-1}/X_{N-2}$$

- Define the minimal monomorphism functor $\mathsf{Mimo} : \mathsf{Mor}_{N-1}(A) \to \mathsf{MMor}_{N-1}(A)$ by

$$\mathsf{Mimo}(X_1 \to \cdots \to X_{N-1}) = X_1 \hookrightarrow X_2 \oplus I_2 \hookrightarrow \cdots \hookrightarrow X_{N-1} \oplus I_{N-1}$$

  for some injective objects I_i.
- Define the **rotation** functor $R : \mathsf{stab}_N(A) \to \mathsf{stab}_N(A)$ to be the composition $\mathsf{Mimo} \circ \mathsf{Cok}$.

The Rotation Functor and Suspension

$$\begin{array}{ccc} \mathrm{stab}_N(A) & \xrightarrow{R} & \mathrm{stab}_N(A) \\ \downarrow \cong & & \downarrow \cong \\ D_N^s(A) & \xrightarrow{\Sigma[-1]} & D_N^s(A) \end{array}$$

- Since $(\Sigma[-1])^N = \Sigma^N[-N] = \Sigma^{N-2}$, we have that $R^N = \Omega_N^{N-2}$.

$$\begin{array}{rcl} X_\bullet & = & 0 \longrightarrow 0 \longrightarrow X \\ R(X_\bullet) & = & X \longrightarrow X \longrightarrow X \\ R^2(X_\bullet) & = & X \hookrightarrow I_X^0 \longrightarrow I_X^0 \\ R^3(X_\bullet) & = & 0 \longrightarrow \Omega^{-1}X \longrightarrow I_X^{-1} \\ R^4(X_\bullet) & = & 0 \longrightarrow 0 \longrightarrow \Omega^{-2}X \end{array}$$

Serre Functor

- Let $\nu_A = D\operatorname{Hom}_A(-, A)$ be the Nakayama functor on $\mathrm{mod}\text{-}A$.
- Let $\nu_{A*} : \mathrm{stab}_N(A) \to \mathrm{stab}_N(A)$ act by ν_A componentwise.
- $S := \Omega_N R \nu_{A*}$ is a Serre functor for $\mathrm{stab}_N(A)$.
- If the Nakayama automorphism of A has order r, let $x = lcm(N, r), y = \frac{x}{N}$. Then, for $N > 2$,

$$S^x = \Omega_N^{2y}$$

- Thus $\mathrm{stab}_N(A)$ is $\frac{-2y}{x}$-Calabi-Yau.
- When A is symmetric, $\mathrm{stab}_N(A)$ is $\frac{-2}{N}$-Calabi-Yau.

Thank you!