Semisimple Homology Resolutions

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(joint, ongoing work w/ Ben Briggs
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A = rt. noeth. ring, $\mathrm{mod}(A)$ = fg rt. A-modules

1. Motivation

Generalize finite projective Resolutions

= complexes $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$

s.t. ① each P_i = f.g. projective

② it is exact.

relax ① $\to$ Resolutions by other modules

eg Representation Dimension

G-dim

Permutation resolutions (Balmer - Gallauer)

noncommutative resolutions

or Relax ② $\to$ nonexact resolutions by projectives

eg Benson - Carlson '87 : Allow semisimple homology

CRH-resolutions

Gheibi, Jorgensen, Takahashi : allow homology in $\mathrm{add}(M)$ $\to$ Quasiprojective Resolutions

2. Definitions, Examples, Questions

<u>Def</u> A <u>semisimple homology (ssh) resolution</u> of $M_\bullet \in K^b(\text{mod}\, A)$ is a map $\varepsilon : P_\bullet \longrightarrow M_\bullet$ s.t.

① $P_\bullet$ perfect

② $H_i(\varepsilon)$ = isomorphism $\forall\, i \leq \sup M$

③ $H_i(P)$ = semisimple $\forall\, i > \sup M$

<u>Examples</u> • $\text{pd}_A M < \infty \Rightarrow$ proj. Res. $0 \to P_n \to \cdots \to P_0 \to M \to 0$ is ssh res of M.

• $P_\bullet$ perfect $\Rightarrow$ $\text{Id} : P_\bullet \to P_\bullet$ = ssh res of $P_\bullet$

• $M_\bullet$ acyclic $\Rightarrow$ $0 \to M_\bullet$ is an ssh res of $M_\bullet$

• $\Omega^n(M)$ = semisimple $\Rightarrow$ $\underbrace{0 \to P_{n-1} \to \cdots \to P_0}_{\text{truncated proj. Res}} \to M \to 0$ is ssh res of M.

• A comm. $\mathfrak{m} = (x_1, \ldots, x_r)$ = max ideal $K(\underline{x})$ koszul complex
$K(\underline{x}) \to A/\mathfrak{m}$ is an ssh resolution.

Natural Questions Existence: For which A, M do ssh resolutions exist?

& what are the consequences for A? for M?

* Is there a f.g. M_A over A = f.d. alg. that does not have an ssh resolution?

Yes!

Structure If M has an ssh resolution

① what is the minimal length of an ssh resolution of M?

② What is the minimal length of the homology in an ssh resolution?

BC conjectured lower bound for ssh resolutions of k over kG (disproved by recent work)
I-walker

Example (Tiago Cruz, after talk) $A = kQ/I$ $Q = 1 \rightleftarrows 2$

w/ proj. indecomps $\begin{smallmatrix}1\\2\\1\\2\end{smallmatrix}$ $\begin{smallmatrix}2\\1\\2\end{smallmatrix}$

The module $\begin{smallmatrix}1\\2\end{smallmatrix}$ does not have an ssh resolution.

In fact: $HR(A) = K^b(\text{proj-}A)$ (see below)

3. Existence Results

k = alg. closed field.

For the following A, every finite length A-module has an ssh-Resolution:

① [B-C] $A = kG$ G = fin. gp. k = field

② A = f.d. k-alg. w/ (F_g) $\left\{ \begin{array}{l} HH^*(A) \text{ is noetherian} \\ + \\ Ext^*(A/J, A/J) \text{ is f.g.} / HH^*(A) \end{array} \right.$

$J = \operatorname{rad} A$.

③ A = f.d self-injective k-alg. w/ $J^3 = 0$.

(many do not satisfy (F_g))

④ A = commutative

↳ cor A = local, Gorenstein isolated singularity, then every f.g. A-module is a direct summand of a module w/ an ssh resolution.

4. Notes on Proofs

General Strategy [BC]: First $\overbrace{\text{find ssh res. of simples}}^{\text{different in each case}}$

Then $\underbrace{\text{"take extensions"}}_{\text{nontrivial, but very general}}$

Def $M_\bullet$ is <u>highly ssh resolvable</u> if $\forall n \geq 0$ $\exists$ ssh res. $\varepsilon: P_\bullet \to M_\bullet$ s.t. $H_i(\mathrm{cone}(\varepsilon)) = 0$ $\forall i \leq n$

$\Big\}$ ie, $P_\bullet$ initially coincides w/ proj. res. of M.

$HR(A) = \{ M_\bullet \in k^b(A) \mid M \text{ is highly ssh resolvable} \} \subseteq k^b(A)$

↳ induces subcats of $D^b(\mathrm{mod}\, A)$ & of $D_{sg}(A)$.

Thm $HR(A)$ = triangulated subcat. of $k^b(\mathrm{mod}\, A)$.
(or $D^b(\mathrm{mod}\, A)$, $D_{sg}(A)$)

But: We don't know/expect it to be closed under summands

Cor If every simple A-module is highly ssh-resolvable then so is every finite length A-module.

② A satisfies (F_g) Let $\zeta_1, \dots, \zeta_d \in HH^*(A)$ be a H.S.O.P. for $\mathrm{Ext}^*_A(A/J, A/J)$

we can control their degrees $d_i = |\zeta_i|$ by using suitable powers

$n < d_1$, $d_i > d_1 + \dots + d_{i-1}$

For S_A simple, set $L_0 = S \in D^b(A)$

Define L_i inductively: $L_i \to L_{i-1} \xrightarrow{\zeta_i} L_{i-1}[d_i] \to$

$L_i = \Sigma^{-1}(L_{i-1} /\!/ \zeta_i)$

$\mathrm{Ext}^*(L_d, L_d)$ annihilated by powers of all ζ_i

$\Rightarrow$ L_d perfect complex.

long exact homology seq. $\Rightarrow$ $\left.\begin{array}{l} H_i(L_d) \in \mathrm{add}(S) \\ H_i(L_d) = 0 \quad \forall\, 0 < i \leq n. \end{array}\right\} S \in \mathrm{IR}(A)$

④ A commutative

$S = A/\mathfrak{m}$ simple $\quad \mathfrak{m} = (x_1, \dots, x_r) \quad \underline{x} = x_1, \dots, x_r$

$K_\cdot = K(\underline{x})$ koszul complex $\quad \leftarrow$ DG-algebra

$K(\underline{x}) \to A/\mathfrak{m}$ ssh Res. $\leftarrow$ How to extend this to ssh resolutions that begin exact.

take a semifree resolution $\varepsilon : F_\cdot \xrightarrow{\simeq} A/\mathfrak{m}$ over K.

$F(n) =$ dg-submodule gen. by basis elements of deg $\leq n$

$\Rightarrow \quad \varepsilon|_{F(n)} : F(n) \longrightarrow A/\mathfrak{m}$ is an ssh resolution that is exact in degs $\leq n$

$\Rightarrow A/\mathfrak{m} \in \mathrm{HR}(A)$

5. Other Interesting Facts.

Prop Assume A is artinian and all f.g. A-modules have ssh resolutions. Then there exits $N \in \mathbb{N}$ s.t. every M_A has an ssh resolution of length $\leq N$

A = Self Injective $\to$ $\underline{\text{mod}}(A)$ triangulated $\Sigma = \Omega^{-1}$. $\mathcal{S} = \text{add}(A/J)$

notation $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T} \Rightarrow \mathcal{A} * \mathcal{B} = \{C \in \mathcal{T} \mid \exists \triangle\ A \to C \to B \to \Sigma A\}$, $A \in \mathcal{A}$, $B \in \mathcal{B}$

Prop The subcat. of $\underline{\text{mod}}(A)$ of all modules with an ssh Res. of length $\leq d$ is:

$$\underbrace{\mathcal{X}_d = \Sigma^{d+1}\mathcal{S} * \cdots * \Sigma^2 \mathcal{S}}$$

$$0 \to S \to P_1 \to P_0 \to M \to 0, \quad M \in \Sigma^2 \mathcal{S}$$

Cor every M_A has ssh resolution $\iff$ $\exists d$

$$\underbrace{\underline{\text{mod}}(A) = \Sigma^{d-1}\mathcal{S} * \cdots * \Sigma\mathcal{S} * \mathcal{S}}$$

$\mathcal{S}$ "shift-generates" $\underline{\text{mod}}(A)$