Singularity categories and Cluster categories

(joint work with Norihiro Hanihara)

Osamu Iyama
(The university of Tokyo)

Cohen-Macaulay representations

R : commu. compl. local Gorenstein ring of dim. d

$\underline{\mathrm{CM}}R$: Frobenius category of max. Cohen-Macaulay mod.

$$(\Leftrightarrow \operatorname{depth} X = d \Leftrightarrow \operatorname{Ext}^{>0}_R(X,R) = 0)$$

[Auslander 76] R : isolated singularity

① $\mathrm{CM}R$ has AR sequences

② $\underline{\mathrm{CM}}R$ is a $(d-1)$-Calabi-Yau triang. cat.

(AR duality)

[Buchweitz] $\underline{\mathrm{CM}}R \simeq D_{sg}(R) := \dfrac{D^b(\operatorname{mod} R)}{\operatorname{per} R}$

- Another class of Calabi-Yau triang cat.

Cluster categories $\mathcal{C}_d(A)$ [BMRRT, Amiot, Guo, Keller ⋯]

<u>Q</u> Connection between singularity cat. and cluster cat.

<u>Ex</u> R : simple singularity of even. dim.

Q : corresponding Dynkin quiver

$$\Rightarrow \quad \underline{\mathrm{CM}}\, R \simeq \mathcal{C}_1(kQ)$$

$$\frac{k[\![x,y,z]\!]}{(x^{n+1}+yz)} \qquad 1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows n$$

<u>Aim</u> Explain a general strategy to construct equivalences between singularity cat. and cluster cat.

Apply it to give new equivalences

$\mathcal{T}$: algebraic triang. cat. A : f.d. alg

(tilting theory) To construct a triangle equiv. $\mathcal{T} \simeq \operatorname{per} A$,

it suffices to find out a tilting obj. $T \in \mathcal{T}$ s.t. $\operatorname{End}(T) \simeq A$

<u>Difficulty</u> To construct a triangle equiv. $\mathcal{T} \simeq \mathcal{C}_d(A)$,

it does not suffice to find out a d-cluster tilting obj.

<u>Our methods</u> Use

① canonical dg enhancement of $\mathcal{C}_d(A)$ equipped with
$\mathbb{Z}$-grading

② tilting theory for $\mathbb{Z}$-graded singularity category

<u>Setting</u> • $R = \bigoplus_{i\geq 0} R_i$: $\mathbb{Z}$-graded Gorenstein isolated sing.

dim d, $R_0 = k$: field. a-invariant $a \neq 0$

(i.e. $\mathrm{Ext}^d_R(k, R(a)) \simeq k$ in $\mathrm{mod}^{\mathbb{Z}} R$)

• A : fin. dim. k-alg

<u>Main Thm</u> Assume $D^{\mathbb{Z}}_{sg}(R) \left(:= \dfrac{D^b(\mathrm{mod}^{\mathbb{Z}} R)}{\mathrm{per}^{\mathbb{Z}} R} \right) \simeq \mathrm{per}\, A$

① $D^{\mathbb{Z}/a\mathbb{Z}}_{sg}(R) \simeq \mathcal{C}_{d-1}(A)$

② $D_{sg}(R) \simeq \mathcal{C}^{(1/a)}_{d-1}(A)$: triang. hull of $\mathrm{per}\, A / F$

for $\exists F$ with $F^a = \nu_0[-d]$

Ex 1 ($d=0$)

$R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$-graded Artinian Gorenstein ring

$R_0 = k$: field. $a =$ degree of $\operatorname{soc} R$

[Yamaura 13] $T := \bigoplus_{i=1}^{a} R(i)_{\geq 0} \in D_{sg}^{\mathbb{Z}}(R)$ is a tilt. obj

$$D_{sg}^{\mathbb{Z}}(R) \simeq \operatorname{per} A \text{ for } A := \operatorname{End}_{D_{sg}^{\mathbb{Z}}(R)}(T) = \begin{bmatrix} R_0 & & & 0 \\ R_1 & R_0 & & \\ & & \ddots & \\ R_{a-1} & R_{a-2} & \cdots & R_0 \end{bmatrix}$$

By Main Thm

$$\boxed{\begin{aligned} D_{sg}^{\mathbb{Z}/a\mathbb{Z}}(R) &\simeq \mathcal{C}_{-1}(A) \\ D_{sg}(R) &\simeq \mathcal{C}_{-1}^{(1/a)}(A) \end{aligned}}$$

[Yamaura 13]

Ex 2 $(d=1)$

$R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$-graded reduced Gorenstein ring of dim 1

$R_0 = k$: field. a : a-invariant of R

[Buchweitz - I - Yamaura 20]

For $n \gg 0$, $T := \bigoplus_{i=1}^{n} R(i)_{\geq 0} \in D_{sg}^{\mathbb{Z}}(R)$ is a tilt. obj

$$D_{sg}^{\mathbb{Z}}(R) \simeq \mathrm{per}\, A \quad \text{for} \quad A := \mathrm{End}_{D_{sg}^{\mathbb{Z}}(R)}(T)$$

By Main Thm

$$\boxed{\begin{aligned} D_{sg}^{\mathbb{Z}/a\mathbb{Z}}(R) &\simeq \mathcal{C}_0(A) \\ D_{sg}(R) &\simeq \mathcal{C}_0^{(1/a)}(A) \end{aligned}}$$

Ex3 (quotient singularity) k : field

$SL_d(k) \supset G$: finite subgroup s.t. $\# G \neq 0$ in k

$G \curvearrowright S := k[x_1, \cdots, x_d]$

$R := S^G$ $a = -d$

$\Big\}$ $\mathbb{Z}$-graded by $\deg x_i = 1$

[I-Takahashi 13]

$T := \bigoplus_{i=1}^{d}$ (max. CM summand of $\Omega_S^i(k)) \in D_{sg}^{\mathbb{Z}}(R)$ is tilt.

$$D_{sg}^{\mathbb{Z}}(R) \simeq \operatorname{per} A \quad \text{for} \quad A := \operatorname{End}_{D_{sg}^{\mathbb{Z}}(R)}(T)$$

By Main Thm

$$\boxed{\begin{aligned} D_{sg}^{\mathbb{Z}/d\mathbb{Z}}(R) &\simeq \mathcal{C}_{d-1}(A) \\ D_{sg}(R) &\simeq \mathcal{C}_{d-1}^{(1/d)}(A) \end{aligned}}$$

Main Thm can be generalized to

- suitable non-commutative rings
- with G-grading for an abelian group

Ex4 (weighted proj. lines, Geigle-Lenzing compl. intersect.)

$S := k[t_1, \cdots, t_d] \ni \ell_1, \cdots, \ell_n$: linear forms, as linear

$\mathbb{Z}_{\geq 1} \ni p_1, \cdots, p_n$ independent as possible

$$R := S[x_1, \cdots, x_n] / (x_i^{p_i} - \ell_i \mid 1 \leq i \leq n)$$

$$\mathbb{L} := \bigoplus_{i=1}^{n} \mathbb{Z}\vec{x_i} \oplus \mathbb{Z}\vec{c} / (p_i\vec{x_i} - \vec{c} \mid 1 \leq i \leq n)$$

R is $\mathbb{L}$-graded by $\deg x_i = \vec{x_i}$, $\deg t_j = \vec{c}$

[Herschend – I – Minamoto – Oppermann, Kussin – Meltzer – Lenzing
Futaki – Ueda]

$\exists A$: f. d. alg. s.t. $D^{\mathbb{L}}_{sg}(R) \simeq \operatorname{per} A$

<u>Ex</u> $n = d+1 \Rightarrow A = \bigotimes_{i=1}^{n} kA_{p_i-1}$

By $\mathbb{L}$-graded version of <u>Main Thm</u>

$$\boxed{D_{sg}^{\mathbb{L}/(\vec{\omega})}(R) \simeq \mathcal{C}_{d-1}(A)}$$

$\vec{\omega} \in \mathbb{L}$: a-invariant

Ex 5 (Grassmannian Cluster categories) $0 < \ell < n$

$D_{sg}^{\mathbb{Z}/n\mathbb{Z}}(R)$ for $R := k[x, y]/(x^{\ell} - y^{n-\ell})$

$\mathbb{Z}/n\mathbb{Z}$- graded by $\deg x = 1$, $\deg y = -1$

By $\mathbb{L}$- graded "hypersurface" version of Main Thm

$$\boxed{D_{sg}^{\mathbb{Z}/n\mathbb{Z}}(R) \simeq \mathcal{C}_2(kA_{\ell-1} \otimes kA_{n-\ell-1})}$$

Key point $R = k[t_1, x_1, x_2]/(x_1^{\ell} - t_1, x_2^{n-\ell} - t_1)$

is a Geigle-Lenzing compl. intersection for $d = 1$, $n = 2$

$p_1 = \ell$, $p_2 = n - \ell$, $\ell_1 = \ell_2 = t_1$

$\mathbb{L} \twoheadrightarrow \mathbb{L}/(\ell\vec{x_1} - (n-\ell)\vec{x_2}) \simeq \mathbb{Z}/n\mathbb{Z}$

- $\mathcal{A}$: dg category / k $\quad \mathcal{A}^e := \mathcal{A}^{op} \otimes \mathcal{A}$

  V : cofibrant dg $\mathcal{A}^e$-module

[Keller 05] dg orbit category $\mathcal{A}/V$ with same obj. as $\mathcal{A}$

$$(\mathcal{A}/V)(X,Y) := \coprod_{n \in \mathbb{Z}} \operatorname*{colim}_{i \gg 0} \mathrm{Hom}_{\mathcal{A}}\left(V^{\otimes_{\mathcal{A}} n+i}(-,X),\ V^{\otimes_{\mathcal{A}} i}(-,Y)\right)$$

with canonical differential and additional $\mathbb{Z}$-grading

$$\begin{array}{ccc}
\text{deg}\,0 & \text{deg}\,{-1} & \text{deg}\,{-2} \\
(\mathcal{A},\mathcal{A}) & (\mathcal{A},V) & (\mathcal{A},V^{\otimes 2}) \\
\text{deg}\,1 & \searrow \quad & \searrow \quad \quad \searrow \\
(V,\mathcal{A}) & (V,V) & (V,V^{\otimes 2}) \quad \ddots \\
\text{deg}\,2 & \searrow \quad & \searrow \quad \quad \searrow \\
(V^{\otimes 2},\mathcal{A}) & (V^{\otimes 2},V) & (V^{\otimes 2},V^{\otimes 2}) \quad \ddots \\
\searrow & \searrow & \searrow
\end{array}$$

Ex A. fin dim. k-alg. gl. dim $A < \infty$ $\quad d \in \mathbb{Z}$

$\mathcal{A} := \mathrm{per}_{dg} A \ (= C^b(\mathrm{proj}\, A)_{dg})$

P: cofibrant resolution of $DA[-d]$

$V(X, Y) := \mathcal{A}(X \otimes_A P, Y)$

$\mathcal{C}_d(A) := \mathrm{per}(\mathcal{A}/V)$: d-cluster category

---

- $\mathcal{B} = (\bigoplus_{i,j \in \mathbb{Z}} \mathcal{B}^i_j, \ d : \mathcal{B}^i_j \longrightarrow \mathcal{B}^{i+1}_j)$: $\mathbb{Z}$-graded dg category

[1] $\circlearrowleft$ $D^{\mathbb{Z}}(\mathcal{B})$: derived cat. of $\mathbb{Z}$-graded dg $\mathcal{B}$-modules

(1) $\circlearrowleft$ $\cup$

$\mathrm{per}^{\mathbb{Z}}(\mathcal{B}) := \mathrm{thick}\{\mathcal{B}(-, X)(i) \mid X \in \mathcal{B}, i \in \mathbb{Z}\}$

Key Prop 1 $\mathcal{B}$: $\mathbb{Z}$-graded dg category

Assume $\mathrm{per}^{\mathbb{Z}}\mathcal{B} = \mathrm{thick}\{\mathcal{B}(-,X) \mid X \in \mathcal{B}\}$

$\mathcal{A} := \mathcal{B}_0$: dg cat. $V := \mathcal{B}_{-1}$: dg $\mathcal{A}^e$-mod.

$\boxed{\text{① } \mathcal{B} \text{ and } \mathcal{A}/V \text{ are } \mathbb{Z}\text{-graded quasi-equivalent}}$

i.e. $\exists$ $\mathbb{Z}$-graded dg $(\mathcal{A}/V)^{op} \otimes \mathcal{B}$-mod. W s.t.

$$-\overset{L}{\underset{\mathcal{A}/V}{\otimes}} W : \mathrm{per}^{\mathbb{Z}}(\mathcal{A}/V) \xrightarrow{\sim} \mathrm{per}^{\mathbb{Z}}\mathcal{B}$$

$$\begin{array}{ccc} \mathrm{per}^{\mathbb{Z}}(\mathcal{A}/V) & \xrightarrow{\sim} & \mathrm{per}^{\mathbb{Z}}\mathcal{B} \\ \cup & & \cup \\ \mathrm{add}\{(\mathcal{A}/V)(-,Y)\}_{Y \in \mathcal{A}/V} & \xrightarrow{\sim} & \mathrm{add}\{\mathcal{B}(-,X)\}_{X \in \mathcal{B}} \end{array}$$

② Assume $\exists d \in \mathbb{Z}$ s.t. $\overset{\substack{k\text{-dual} \\ \downarrow}}{D} B \simeq B(-1)[d]$ in $D(B^e)$

Then $\boxed{\begin{array}{c}\exists \; \begin{array}{ccc} \mathrm{per}_{dg}\, \mathcal{A} & \xrightarrow{\sim} & \mathrm{per}_{dg}^{\mathbb{Z}} B \quad : \text{ iso. in Hqe} \\ \downarrow & \circlearrowright & \downarrow \\ \mathrm{per}_{dg}(\mathcal{A}/D\mathcal{A}[-d]) & \xrightarrow{\sim} & \mathrm{per}_{dg} B \quad : \text{ iso in Hqe} \end{array}\end{array}}$

$$\mathrm{Hqe} := \mathrm{dgcat}_k \left[\{\text{quasi-equiv.}\}^{-1} \right]$$

<u>Def</u> R : Gorenstein ring $\mathbb{Z}$-graded

$$D_{sg}^{\mathbb{Z}}(R)_{dg} := \{ X \in C(\mathrm{proj}^{\mathbb{Z}} R)_{dg} \mid H(X) = 0 \}$$

dg singularity category

Key Prop 2 (enhanced AR duality)

$R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$- graded Gorenstein isolated sing.

dim d , $R_0 = k$: field. a- invariant a

$\mathcal{C} := D^{\mathbb{Z}}_{sg}(R)_{dg}$ $\quad ( )^* = \mathrm{RHom}_R(-, R)$

$\Rightarrow \exists$ iso. $\left.\begin{array}{l} \mathcal{C}^* \simeq \mathcal{C}[-1] \\ D\mathcal{C} \simeq \mathcal{C}(-a)[d-1] \end{array}\right\}$ in $D^{\mathbb{Z}}(\mathcal{C}^e)$

Taking H^0, we recover classical AR duality

$$\underline{\mathrm{Hom}}^{\mathbb{Z}}_R(X, Y) \simeq D\,\underline{\mathrm{Hom}}^{\mathbb{Z}}_R(Y, X(a)[d-1])$$

<u>Sketch</u>

$$\overset{\mathcal{A}}{\overset{\shortparallel}{C^b(\mathrm{proj}^{\mathbb{Z}}R)_{dg}}} \to \overset{\mathcal{B}}{\overset{\shortparallel}{C^{-,b}(\mathrm{proj}^{\mathbb{Z}}R)_{dg}}} \xrightarrow[\exists \text{ quasi-functor}]{V} \mathcal{C}$$

s.t. $0 \longrightarrow D(\mathcal{A}) \xrightarrow{- \overset{L}{\otimes}_{\mathcal{A}} \mathcal{B}} D(\mathcal{B}) \xrightarrow{- \overset{L}{\otimes}_{\mathcal{B}} V} D(\mathcal{C}) \longrightarrow 0$: exact

$\mathcal{B} \overset{L}{\otimes}_{\mathcal{A}} \mathcal{B} \longrightarrow \mathcal{B} \longrightarrow \mathrm{REnd}_{\mathcal{C}}(V) \longrightarrow$: triangle in $D(\mathcal{B}^e)$

Evaluate $X, Y \in \mathcal{B}$

Switch X, Y Apply $(\ )^*$

$$\begin{tikzcd}
Y \otimes_R X^* \arrow[r, "\alpha_{XY}"] \arrow[d, "\wr"] & \mathrm{Hom}_R(X,Y) \arrow[r] \arrow[d, "\wr"] & \mathcal{C}(X,Y) \arrow[r] \arrow[d, "\wr"] & {} \\
\mathrm{Hom}_R(Y,X)^* \arrow[r, "\alpha_{YX}^*"] & (X \otimes_R Y^*)^* \arrow[r] & \mathcal{C}(Y,X)^*[1] \arrow[r] & \square
\end{tikzcd}$$

- <u>Key Prop 1</u> ② + (enhanced AR duality)

$\Longrightarrow$ <u>Main Thm</u> for $a = 1$

<u>Problem</u> Can we deduce all known equivalences between singularity categories and cluster categories from our results (and generalization)?

- [Amiot-Reiten-Todorov] Π : preprojective alg. $w \in W$

  $$D_{sg}(\Pi_w) \simeq \mathcal{C}_2(Q, W)$$

- [I-Oppermann] Π : higher preproj. alg. of d-rep.-fin. alg. A

  $$D_{sg}(\Pi) \simeq \mathcal{C}_{d+1}(\underline{d\text{-Aus}}(A))$$

- [Amiot-I-Reiten] Π : higher preproj. alg. of d-rep.-infin. alg. A

  $$D_{sg}(e\Pi e) \simeq \mathcal{C}_d(A/(e))$$

  $A \ni e$: certain idemp.