Cutting surfaces and recollements of gentle algebras

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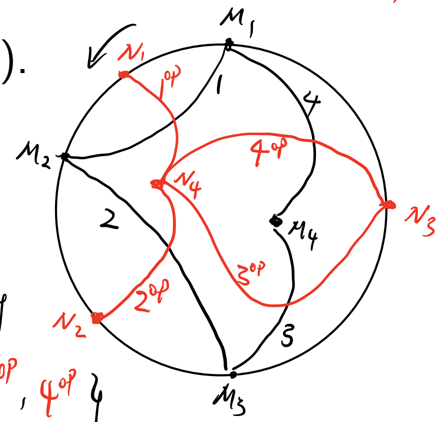
Setting

- S: compact oriented surface with boundary $\partial S := \bigcup_i \partial_i S$.

$M_\bullet = \{M_1, M_2, M_3, M_4\}$
$M_{\color{red}\bullet} = \{{\color{red}N_1, N_2, N_3, N_4}\}$

- $M = \underbrace{M_\bullet \sqcup M_{\color{red}\bullet}}$: marked points (on ∂S or $S \backslash \partial S$).

  ↳ alternate on $\partial_i S$



- Δ: graded admissible $\underline{\bullet\text{-dissection}}$ of S.

  into polygons

$\Delta = \{1, 2, 3, 4\}$

- ${\color{red}\Delta^*}$: dual ${\color{red}\bullet}$-dissection of Δ.

${\color{red}\Delta^*} = \{{\color{red}1^{op}, 2^{op}, 3^{op}, 4^{op}}\}$

Fact (Opper-Plamondon-Schroll'18, Palu-Pilaud-Plamondon'18)

$$\{\textit{graded surface dissections}\}/\text{homeo.} \xleftrightarrow{1:1} \{\textit{graded (locally) gentle algebras}\}/\text{iso.}$$

$$(S, M, \Delta) \longleftarrow\longrightarrow A(\Delta)$$

Main result

Theorem (Chang-Jin-Schroll)

Let (S, M, Δ) be a graded surface dissection. Let $\Gamma \subset \Delta$. Then there is a recollement of derived category of dg gentle algebras

$$\begin{tikzcd} \mathcal{D}(A(\Delta_\Gamma)) \arrow[r] & \mathcal{D}(A(\Delta)) \arrow[l, shift right=3] \arrow[l, shift left=3] \arrow[r] & \mathcal{D}(A(\Delta^*_{(\Delta^*\setminus\Gamma^*)})^*) \arrow[l, shift right=3] \arrow[l, shift left=3] \end{tikzcd}$$

where

- $A(\Delta_\Gamma)$ *is obtained by cutting* (S, M, Δ) *along* Γ.
- $A(\Delta^*_{(\Delta^*\setminus\Gamma^*)})^*$ *is obtained by cutting* (S, M, Δ^*) *along* $\Delta^*\setminus\Gamma^*$.

Graded quadratic monomial algebras

- k: field
- Q: graded quiver with $Q_0 := \{1, 2, \dots, n\}$ and a map $|\cdot| : Q_1 \to \mathbb{Z}$.
- I: a set of quadratic monomial relations ($I = \{\alpha\beta, \gamma\delta, \cdots\}$).
- $A = kQ/\langle I \rangle$: dg k-algebra with 0 differential.
- $e = e_1 + e_2 + \cdots + e_m$: idempotent of A.
- $J := \{\alpha\beta \in I \mid t(\alpha) = s(\beta) \in \{1, 2, \dots, m\}\} \subset I$.
- $J_1 := Q_1$ and $J_n := \{\alpha_1 \cdots \alpha_n \mid \alpha_i\alpha_{i+1} \in J, 1 \leq i \leq n-1\}$ for $n \geq 2$.

$J_2 = J \qquad J_3 = \{\alpha\beta\gamma \mid \alpha\beta \in J, \beta\gamma \in J\}$

Graded quadratic monomial algebras

Theorem (Chang-Jin-Schroll)

There is a recollement of derived categories

$$\mathcal{D}(A_e) \rightleftarrows \mathcal{D}(A) \rightleftarrows \mathcal{D}(eAe) \qquad (1)$$

where $A_e = (kQ'/I', d')$ *is given by*

- $(Q')_0 := Q_0 \backslash \{1, 2, \ldots, m\}$.
- *The arrows of* Q' *are of the form* $[\alpha_1 \cdots \alpha_s] : s(\alpha_1) \to t(\alpha_s)$ *with* $\alpha_1 \cdots \alpha_s \in J_s$ *and* $s(\alpha_1), t(\alpha_s) \notin \{1, 2, \ldots, m\}$.
- $|[\alpha_1 \cdots \alpha_s]| := \sum_{i=1}^{s} |\alpha_i| - s + 1$.
- $I' := \{[\alpha_1 \ldots \alpha_s][\beta_1 \ldots \beta_t] \mid \alpha_s \beta_1 \in I\}$.
- $d' = 0$.

Remark

- If $J = \emptyset$, then $A_e = A/AeA$ and (1) is well-known for ungraded algebra A [Cline-Parshall'96].
- See [Drinfeld'04][Nicolás-Saorín'09][Kalck-Yang'16,'18]...for other constructions.

Fact

A_e is also a graded quadratic monomial algebra. Moreover, if A is a graded gentle algebra, so is A_e.

- We define $A^* := kQ^{\mathrm{op}}/\langle I^{\perp}\rangle$ with $|\alpha^{\mathrm{op}}| := 1 - |\alpha|$. "Koszul dual"

$\{\beta^{\mathrm{op}}\alpha^{\mathrm{op}} \mid \alpha\beta \notin I\}$

Proposition

There are isomorphisms of dg k-algebras

- $eAe \cong ((A^*)_{1-e})^*$
- $A_e \cong ((1-e)A^*(1-e))^*$

Example 1. $A: 1 \xrightarrow[a]{\alpha} 2 \xrightarrow[b]{\beta} 3 \quad I=\{\alpha\beta\} \quad e=e_2 \quad \Longrightarrow \quad J=I.$

$J_1=\{\alpha, \beta\}, \quad J_2=\{\alpha\beta\}, \quad J_{\geq 3}=\emptyset.$

$\Longrightarrow \quad A_e: 1 \xrightarrow[a+b-1]{[\alpha\beta]} 3 \qquad eAe=k$

$\Longrightarrow$ Recollement: $\quad \mathcal{D}(1 \xrightarrow[a+b-1]{} 3) \equiv \mathcal{D}(1 \xrightarrow[a]{} 2 \xrightarrow[b]{} 3) \equiv \mathcal{D}(k)$

Example 2. $A: \begin{tikzcd} 1 \arrow[r, "\alpha", "a"'] & 2 \arrow[loop above, "\beta", "b"'] \arrow[r, "\gamma", "c"'] & 3 \end{tikzcd} \quad I=\{\alpha\beta, \beta^2, \beta\gamma\}, \quad e=e_2 \Rightarrow J=I.$

$J_1=\{\alpha, \beta, \gamma\}, \quad J_2=\{\alpha\beta, \beta^2, \beta\gamma\}, \quad J_3=\{\alpha\beta^2, \alpha\beta\gamma, \beta^3, \beta^2\gamma\}$

$\vdots$

$J_n=\{\alpha\beta^{n-1}, \alpha\beta^{n-2}\gamma, \beta^n, \beta^{n-1}\gamma\} \quad n\geq 3.$

$\Longrightarrow \quad A_e: 1 \begin{array}{c} \vdots \\ \xrightarrow[a+3b+c-4]{[\alpha\beta^3\gamma]} \\ \xrightarrow[a+2b+c-3]{[\alpha\beta^2\gamma]} \\ \xrightarrow[a+b+c-2]{[\alpha\beta\gamma]} \end{array} 3$ infinite arrows!

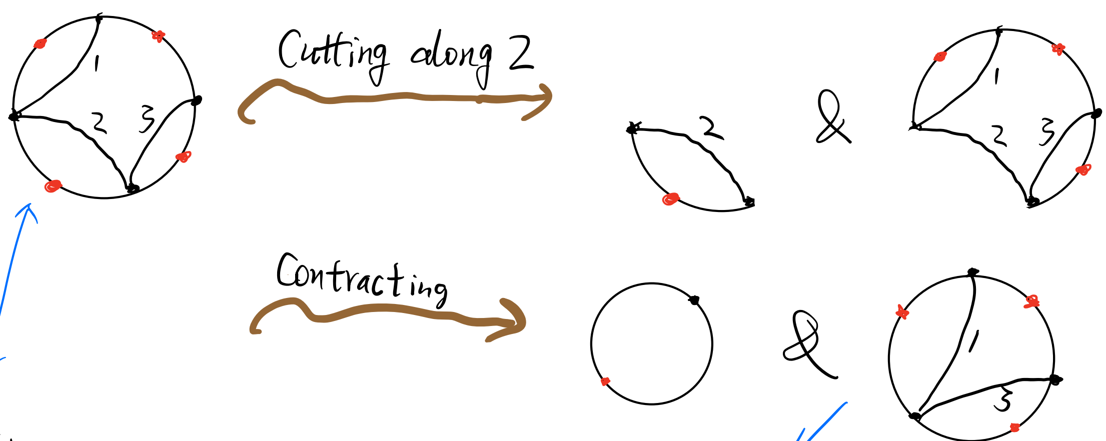
$\Longrightarrow$ Recollement $\quad \mathcal{D}\left(\cdot \begin{array}{c} \vdots \\ \xrightarrow{a+4b+c-5} \\ \xrightarrow{a+3b+c-4} \\ \xrightarrow{a+2b+c-3} \\ \xrightarrow{a+b+c-2} \end{array} \cdot\right) \equiv \mathcal{D}\left(\begin{tikzcd} \cdot \arrow[r, "a"'] & \cdot \arrow[loop above, "b"] \arrow[r, "c"'] & \cdot \end{tikzcd}\right) \equiv \mathcal{D}\left(\begin{tikzcd} \cdot \arrow[loop right, "b"] \end{tikzcd}\right)$

Cutting surface

Definition

Let (S, M, Δ) be a surface dissection and let $\Gamma \in \Delta$. The cut surface along Γ $(S_\Gamma, M_\Gamma, \Delta_\Gamma)$ is obtain by cutting S along Γ and contracting.

Example $\Delta = \{1, 2, 3\}$ $\Gamma = \{2\}$



$A(\Delta) = 1 \to 2 \to 3$

$A(\Delta_\Gamma) = 1 \to 3 \cong A(\Delta)_{e_2}$

Recollement of cutting surface

Proposition

Let (S, M, Δ) be a graded surface dissection and let $\Gamma \subset \Delta$. We may assume $\Gamma = \{1, 2, \ldots, m\}$. Then $(S_\Gamma, M_\Gamma, \Delta_\Gamma)$ is a model of $A(\Delta)_e$, where $e = e_1 + \cdots + e_m$.

Theorem (Chang-Jin-Schroll)

Let (S, M, Δ) be a graded surface dissection. Let $\Gamma \subset \Delta$. Then there is a recollement of derived category of dg gentle algebras

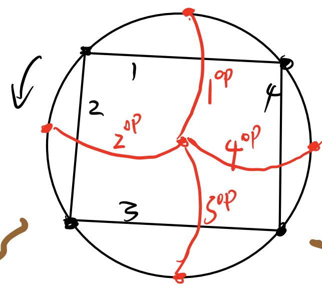
$$\mathcal{D}(A(\Delta_\Gamma)) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} \mathcal{D}(A(\Delta)) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} \mathcal{D}(A(\Delta^*_{(\Delta^* \setminus \Gamma^*)})^*).$$

Example

$\Delta=\{1,2,3,4\}$ $\qquad$ $\Gamma=\{1,2\}$

$\Delta^*=\{1^{op},2^{op},3^{op},4^{op}\}$

$\Gamma^{op}=\{1^{op},2^{op}\}$



Δ_Γ $\qquad$ $\Delta^*_{(\Delta^*\setminus\Gamma^*)}$

$$A(\Delta)= \begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 4 \arrow[u] & 3 \arrow[l] \end{tikzcd}$$

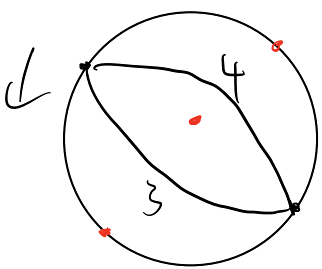
Fact

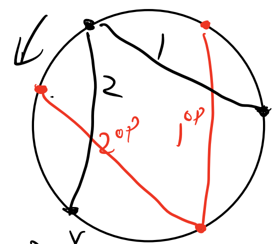
$(S,M,\Delta) \longleftrightarrow A(\Delta)$

$\updownarrow$ dual

$(S,M,\Delta^*) \longleftrightarrow A^*(\Delta)$

$e=e_1+e_2$





$A(\Delta_\Gamma)= 4 \rightleftarrows 3 = A_e$

$A(\Delta^*_{(\Delta^*\setminus\Gamma^*)})^* = 1 \longrightarrow 2 = eAe$