Derived Picard groups of representation-finite symmetric algebras

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modules = right modules

$k=\bar{k}$, A, B fin.-dim k-algebras.

$\mathcal{D}^b(\text{mod-}A)$
$\quad /\!/$

Def. A and B are derived equivalent $\iff \mathcal{D}^b(A) \cong \mathcal{D}^b(B)$ as Δ'ed cat.

tilting complex

[Rickard, Bondal] $\iff \exists\, T \in K^b(\text{proj-}A)$ s.t. $\text{thick}(T) = K^b(\text{proj-}A)$, $\text{End}_{\mathcal{D}^b(A)}(T) \cong B$, $\text{Hom}_{\mathcal{D}^b(A)}(T, T[i]) = 0$ $i \neq 0$.

2-sided tilting complexes, $-\otimes^L X$ "standard equivalences"

[Rickard] $\iff \exists$ bdd complex X of $B^{op}\otimes A$-modules, whose restrictions to A and B^{op} are perfect
$\qquad\qquad\qquad\qquad\quad Y$ of $A^{op}\otimes B$-modules $\qquad\qquad\qquad\qquad\qquad\qquad\quad B \quad\; A^{op}$

$$Y \otimes^L_B X \cong A \text{ in } \mathcal{D}^b(A^{op}\otimes A),\quad X \otimes^L_A Y \cong B \text{ in } \mathcal{D}^b(B^{op}\otimes B)$$

Def. The derived Picard group of A:

$$\text{TrPic}(A) = \{\text{2-sided tilting complexes in } \mathcal{D}^b(A^{op}\otimes A)\}/_{\cong} = \{\text{stand. autoeq. of } \mathcal{D}^b(A)\}/_{\cong}$$

Def. A is symmetric if $A \cong \mathrm{Hom}_k(A,k)$ as bimodules

e.g. group algebras of finite grps, Brauer graph algebras...

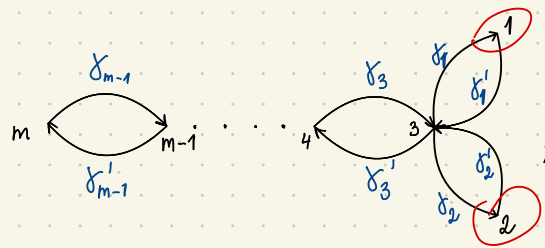
[Asashiba]: $\{\text{basic symmetric rep.-finite alg.}\} \big/ \underset{\cong}{\text{derived}} \xleftrightarrow{1-1} \left\{ \begin{matrix}\text{ADE Dynkin} \\ \text{diagrams}\end{matrix} + \begin{matrix}\text{some} \\ \text{data}\end{matrix} \right\}$

- type A_m = Brauer tree algebras.

  TrPic described by Zvonareva, Volkov-Zvonareva.

- type D_m: one series is algebras derived equivalent to $\Lambda_m = kQ_m/I$

  $\underbrace{m \geq 4}$

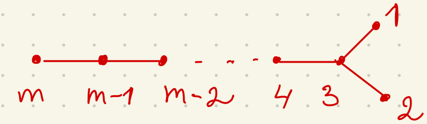
$Q_m =$ 

,

$$I = \left\langle \begin{matrix} \gamma_i\gamma_j,\ \gamma_i'\gamma_j' \quad \forall\, i,j \\ \gamma_k\gamma_k' - \gamma_{k-1}\gamma_{k-1}' \quad \forall\ 4 \leq k \leq m-1 \\ \gamma_1'\gamma_1 - \gamma_3\gamma_3',\ \gamma_2'\gamma_2 - \gamma_3\gamma_3' \end{matrix} \right\rangle$$

Def The braid group of type $\mathcal{D}_m$

$\mathcal{D}_m$



$$B_{\mathcal{D}_m} = \left\langle \{\sigma_i\}_1^m \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } \overset{i\quad j}{\bullet\!\!-\!\!\bullet} \text{ in } \mathcal{D}_m \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{otherwise} \end{array} \right\rangle$$

Theorem [N.] Let $m \geq 5$.

$$G_m = \left\langle \underset{1\leq i\leq m}{\sigma_i \in B_{\mathcal{D}_m}},\ \{a\}_{a\in k^*},\ s,\ \tau \;\middle|\; \begin{array}{l} \tau^2 = e,\quad \sigma_1\tau = \tau\sigma_2,\ \underset{}{\overset{i\neq 1,2}{\tau\sigma_i = \sigma_i\tau}} \\ s \text{ and } \forall a\in k^* \text{ are central} \end{array} \right\rangle$$

Let $c = \sigma_1 \cdots \sigma_m \in B_{\mathcal{D}_m}$.

Then
$$\mathrm{TrPic}(\Lambda_m) \cong G_m \big/ \langle (-1)\, c^{m-1} s^{-2m+3} \rangle \quad \text{if } m \text{ even}$$
$$\cong G_m \big/ \langle (-1)\, c^{m-1} s^{-2m+3} \tau \rangle \quad \text{if } m \text{ odd}$$

$$\begin{tikzcd}[column sep=small]
 & \mathrm{TrPic}(\Lambda_m) & \\
\mathbb{Z} = \langle s \rangle \arrow[ur, hook] & \underbrace{k^*}_{\substack{\| \\ \mathrm{Out}^0(\Lambda_m)}} \times \overbrace{\mathbb{Z}/2\mathbb{Z}}^{e_1 \leftrightarrow e_2} = \langle \{a\}_{a\in k^*}, \tau \rangle \arrow[u, hook] & B_{\mathcal{D}_m} = \langle \sigma_i,\ 1\le i\le m \rangle \arrow[ul, hook'] \\
\text{shift on } \mathcal{D}(\Lambda_m) \arrow[u, leftrightarrow] & \text{outer automorphisms} \arrow[u, leftrightarrow] & \text{spherical twists} \arrow[u, leftrightarrow]
\end{tikzcd}$$

t_{P_i}, $P_i = e_i\Lambda_m$

$P_i = e_i\Lambda_m$, $1 \le i \le m$ are <u>0-spherical objects</u> in $\mathcal{D}(\Lambda_m)$.

- $\mathrm{End}^*_{\mathcal{D}(\Lambda_m)}(P_i) \cong k[t]/t^2$ $\quad (\deg(t) = 0)$

- P_i is 0-Calabi-Yau :

$$\mathrm{Hom}^*_{\mathcal{D}^b(\Lambda_m)}(P_i, -) \cong \mathrm{Hom}^*_{\mathcal{D}^b(\Lambda_m)}(-, P_i)^{\vee}$$

k-dual.

Λ_m is symmetric.

$\rightsquigarrow$ <u>the spherical twist</u>

$$t_{P_i} : \mathcal{D}^b(\Lambda_m) \longrightarrow \mathcal{D}^b(\Lambda_m)$$

$$t_{P_i}(-) = - \otimes^L \underbrace{\mathrm{cone}\left(P_i \otimes P_i^{\vee} \xrightarrow{m} \Lambda_m\right)}_{\text{2-sided tilting complex}}$$

$$e_i a \otimes b e_i \longmapsto ab$$

$\Longrightarrow t_{P_i} \in \mathrm{TrPic}(\Lambda_m)$

$\{P_i\}_1^m$ form a $\mathcal{D}_m$-configuration in $\mathcal{D}^b(\Lambda_m)$:

$$\dim_k \operatorname{Hom}^*_{\mathcal{D}^b(\Lambda_m)}(P_i, P_j) = 1 \quad \text{if } \overset{i}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{j}{\bullet}$$

$$= 0 \quad \text{otherwise}$$

[Seidel-Thomas]

$$\begin{aligned} B_{\mathcal{D}_m} &\xrightarrow{F} \operatorname{TrPic}(\Lambda_m) \\ \sigma_i &\longmapsto t_{P_i} \end{aligned}$$

is a group homomorphism

[N.-Volkov]

F is injective

Remains to

(1) understand the relations between different "blocks" ✓

(2) show that TrPic is generated by such elements:

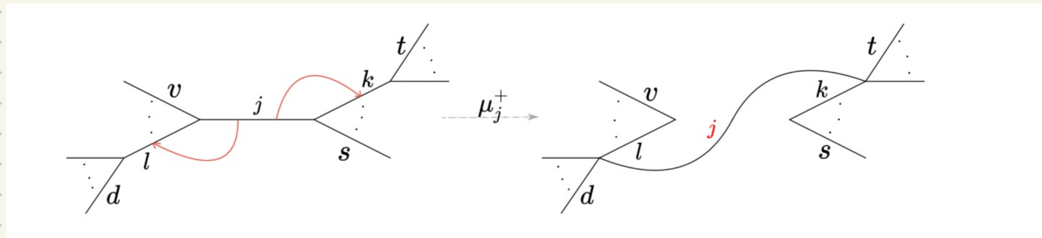
[$T \in K^b(\text{proj-}\Lambda_m)$, $\text{End}_{\mathcal{D}^b(\Lambda)}(T) \cong \Lambda_m \Rightarrow T =$ the image of Λ_m
tilting. under a composition of twists, modulo Out and [1].]

Theorem [Aihara] Λ symmetric rep-finite
$\Rightarrow$ any tilting complex $T \in K^b(\text{proj-}\Lambda)$ concentrated in ≤ 0 degrees can be obtained from Λ via a sequence of silting mutations.

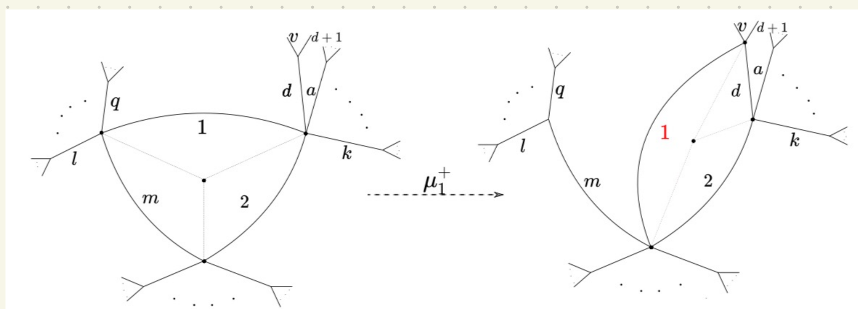
$$\Lambda \text{ symmetric} \Longrightarrow \underset{\text{tilting}}{T = M \oplus X} \overset{\mu_X^+}{\rightsquigarrow} \underset{\text{tilting}}{T' = M \oplus X'}$$

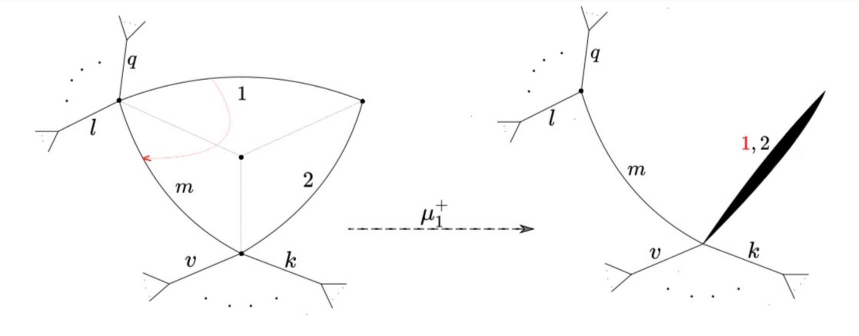
$$\mathrm{End}_{\mathcal{D}^b(A)}(T) \rightsquigarrow \mathrm{End}_{\mathcal{D}^b(A)}(T')$$

For type A = Brauer tree algebras —— classical:



For type D ...




q
1
l
m
2
v
k
μ_1^+
q
l
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1, 2
v
k