The Rigidity of Infinite Frameworks in Euclidean and Polyhedral Normed Spaces

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Normed spaces

A *real normed space* is a vector space X over $\mathbb{R}$ together with a map $\|\cdot\| : X \to [0, \infty)$ such that for all $x, y \in X$ and $\lambda \in \mathbb{R}$:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$.

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For a centrally symmetric polytope $\mathcal{P} \subseteq \mathbb{R}^d$ with with facets $\pm F_1, \ldots, \pm F_n$ we can define the norm $\|\cdot\|_{\mathcal{P}}$ on $\mathbb{R}^d$ by

$$\|x\|_{\mathcal{P}} = \max_{1 \leq k \leq n} \left| \left\langle \hat{F}_k, x \right\rangle \right|$$

where $\hat{F} \in \mathbb{R}^d$ is the unique vector that defines the hyperspace that the face F lies on.

Asimow-Roth for normed spaces

The following is a famous result from *The Rigidity of Graphs* by L.Asimow and B. Roth and an equivalent result for polyhedral normed spaces from *Finite and Infinitesimal Rigidity with Polyhedral Norms* by Derek Kitson.

Theorem

Let (G, p) be a finite, affinely spanning and regular framework in $(\mathbb{R}^d, \|\cdot\|_2)$ or $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$. Then TFAE:

- *(G, p) is infinitesimally rigid*
- *(G, p) is continuously rigid (all deformations are rigid motions)*
- *(G, p) is locally rigid (all equivalent frameworks within a neighbourhood of p are congruent).*

What would be an equivalent result for infinite frameworks in either space?

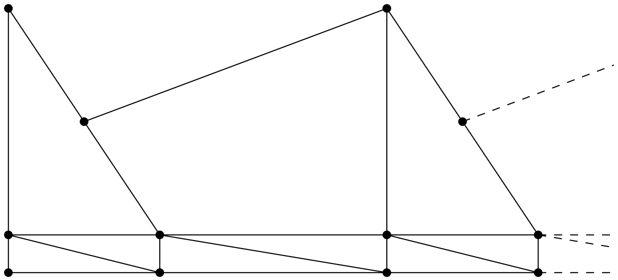

Figure: Infinitesimally rigid but continuously flexible in $(\mathbb{R}^2, \|\cdot\|_2)$. This framework is infinitesimally flexible for all generic positions.

Frameworks

We shall always assume that $(X, \|\cdot\|)$ is a finite dimensional real normed space with an open set of smooth points.

Definition

A *framework* in $(X, \|\cdot\|)$ is a pair (G, p) where G is a simple graph (i.e. no loops, repeated edges and undirected) and $p \in X^{V(G)}$.

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For a framework we will define the *rigidity map* to be

$$f_G : X^{V(G)} \to \mathbb{R}^{E(G)}, \ (x_v)_{v \in V(G)} \mapsto (\|x_v - x_w\|)_{vw \in E(G)}.$$

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Definition

We say an edge $vw \in E(G)$ of (G, p) is *well-positioned* if $p_v - p_w$ is a smooth point and we say (G, p) is *well-positioned* if all edges (G, p) are well-positioned.

Support functionals

For a well-positioned edge $vw \in E(G)$ we define the linear functional $\varphi_{v,w} : X \to \mathbb{R}$ to be the *support functional* of $p_v - p_w$.

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For polyhedral normed space $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$:

$$\varphi_{v,w}(\cdot) = \left\langle \hat{F}, \quad \cdot \quad \right\rangle$$

where $\|p_v - p_w\|_{\mathcal{P}} = \left\langle \hat{F}, p_v - p_w \right\rangle$.

Notation

The space of infinitesimal flexes:

$$\mathcal{F}(G,p) = \left\{ u \in X^{V(G)} : \varphi_{v,w}(u_v - u_w) = 0 \text{ for all } vw \in E(G) \right\}$$

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Orbit of p:

$$\mathcal{O}_p := \{ (h(p_v))_{v \in V(G)} : h \text{ is an isometry of } (X, \|\cdot\|) \}$$

Equicontinuity

Let F be a family of continuous curves $f : I \to X$ for some interval I and some normed space X. We say that F is *equicontinuous at* $t_0 \in I$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$t \in (-\delta + t_0, \delta + t_0) \;\Rightarrow\; \|f(t_0) - f(t)\| < \epsilon$$

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Definition

We say that a family $\alpha = (\alpha_v)_{v \in V(G)}$ of continuous paths $\alpha_v : (-1, 1) \to X$ is an *equicontinuous finite flex* of (G, p) in $(X, \|\cdot\|)$ if:

- $\alpha_v(0) = p_v$ for all $v \in V(G)$
- $\|\alpha_v(t) - \alpha_w(t)\| = \|p_v - p_w\|$ for all $vw \in E(G)$ and $t \in (-1, 1)$
- α is equicontinuous.

Topology of $X^{V(G)}$

For $X^{V(G)}$ we define the *generalised metric* (i.e. a metric that allows infinite distances between points) $d_{V(G)}$ where

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We now define for all $p \in X^{V(G)}$ and $r > 0$ the open balls of $X^{V(G)}$

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For more information on generalised metric spaces see *A Course in Metric Geometry* by Dmitri Burago, Yuri Burago and Sergei Ivanov.

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A framework (G, p) is *locally rigid* (with respect to the $d_{V(G)}$-topology on $X^V(G)$) if there exists $r > 0$ such that $f_G^{-1}[f_G(p)] \cap B_r(p) = \mathcal{O}_p \cap B_r(p)$.

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Definition

A framework (G, p) is *equicontinuously rigid* if all equicontinuous finite flexes are rigid body motions.

Local rigidity implies equicontinuous rigidity

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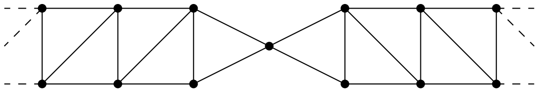

Figure: Locally and equicontinously rigid but infinitesimally and continuously flexible in $(\mathbb{R}^2, \|\cdot\|_2)$.

Bounded infinitesimal rigidity

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Definition

We say that a well-positioned framework (G,p) is *bounded infinitesimally rigid* if $b\mathcal{F}(G,p) \subseteq \mathcal{T}(p)$.

Equivalence of rigidity for Euclidean spaces

Theorem

Let (G, p) be an affinely spanning framework in a d-dimensional Euclidean space such that

- *The points of the placement p are uniformly discrete in X*
- *for some $r > 0$ we have that $b\mathcal{F}(G, q)$ is linearly isomorphic to $b\mathcal{F}(G, p)$ for all $q \in B_r(p)$;*

then the following are equivalent:

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It is an open question whether there is any way of choosing placements such that the condition on linear isomorphisms of bounded flex spaces on an open neighbourhood is automatic.

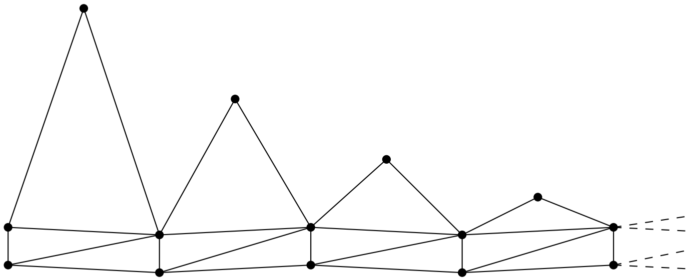

Figure: A generic framework in $(\mathbb{R}^2, \|\cdot\|_2)$ that is infinitesimally and continuously rigid but locally flexible.

Equivalence of rigidity for polyhedral normed spaces

Definition

We say a framework (G, p) is *uniformly well-positioned* if there exists $r > 0$ such that (G, q) is well-positioned for all $q \in B_r(p)$.

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The result is important as checking if a framework is uniformly well-positioned is much easier than checking if all frameworks in a neighbourhood of a placement are bounded infinitesimally rigid.

Special case: $(\mathbb{R}^d, \|\cdot\|_\infty)$

The max norm $\|\cdot\|_\infty$ on $\mathbb{R}^d$:

$$\|(a_1,\dots,a_d)\|_\infty := \max_{1\leq k\leq d} |a_k| = \max_{1\leq k\leq d} |\langle e_k, (a_1,\dots,a_d)\rangle|$$

where $e_1,\dots,e_d$ is the standard basis of $\mathbb{R}^d$.

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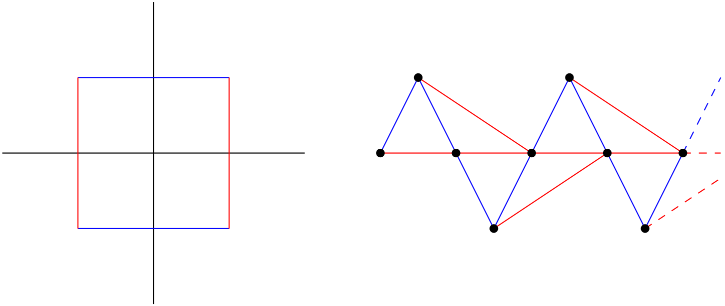

Figure: (Left) Unit ball of $(\mathbb{R}^2, \|\cdot\|_\infty)$; (right) a framework in $(\mathbb{R}^2, \|\cdot\|_\infty)$ that is infinitesimally, equicontinuously and locally rigid.

Thank you for listening!

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Questions?