Sufficient conditions for the global rigidity of periodic graphs

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Definitions

- A graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ is *k-periodic* if there is a subgroup Γ of $\mathrm{Aut}(\widetilde{G})$ isomorphic to $\mathbb{Z}^k$ acting without loops on each vertex of G.
- Γ-labeled graph: $(G = (V, E), \psi)$ with reference orientation $\overrightarrow{E}$ and $\psi : \overrightarrow{E} \to \Gamma$. (Here: NO loops.)
- $G \mapsto \widetilde{G} : \widetilde{V} = \{\gamma v_i : v_i \in V, \gamma \in \Gamma\}$, $\widetilde{E} = \{\{\gamma v_i, \psi(v_i v_j)\gamma v_j\} : (v_i, v_j) \in \overrightarrow{E}, \gamma \in \Gamma\}$.
- For a nonsingular homomorphism $L : \Gamma \to \mathbb{R}^d$ and $p : \widetilde{V} \to \mathbb{R}^d$, $(\widetilde{G}, \widetilde{p})$ is an *L-periodic* framework if

$$\widetilde{p}(v) + L(\gamma) = \widetilde{p}(\gamma v) \qquad \text{for all } \gamma \in \Gamma \text{ and all } v \in \widetilde{V}. \tag{1}$$

- By (1): it is enough to realize G (with $p : V \to \mathbb{R}^d$) and L.
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Theorem (Tanigawa (2016))

For a generically rigid graph $G = (V, E)$, assume that $G - v$ is generically rigid in $\mathbb{R}^d$ and $G - v + K(N_G(v))$ is globally rigid in $\mathbb{R}^d$ for a vertex $v \in V$ with $d(v) \geq d + 1$. Then G is globally rigid in $\mathbb{R}^d$.

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Assume that G is 2-rigid in $\mathbb{R}^d$ Then G is globally rigid in $\mathbb{R}^d$.

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First we extend a Lemma by Bezdek and Connelly (2002).

Lemma

Let $\widetilde{p}$ and $\widetilde{q}$ be two equivalent L-periodic realizations of $\widetilde{G}$ in $\mathbb{R}^d$. Then there exists an $(L, 0^d)$-periodically rigid motion from $(\widetilde{(G)}, (\widetilde{p}, 0^d))$ to $(\widetilde{(G)}, (\widetilde{q}, 0^d))$ in $\mathbb{R}^{2d}$, as follows. Move a vertex γv (for $v \in V$ and $\gamma \in \Gamma$) on the curve $p_{\gamma,v} : [0,1] \to \mathbb{R}^{2d}$ defined by

$$p_{\gamma,v}(t) = \left(\frac{p_{\gamma,v} + q_{\gamma,v}}{2} + (\cos(\pi t)) \frac{p_{\gamma,v} - q_{\gamma,v}}{2}, (\sin(\pi t)) \frac{p_{\gamma,v} - q_{\gamma,v}}{2} \right)$$

where $p_{\gamma,v} = \widetilde{p}(\gamma v)$ and $q_{\gamma,v} = \widetilde{q}(\gamma v)$.

Theorem

If a Γ-labeled framework (G, ψ, p) is not L-periodically globally rigid in $\mathbb{R}^d$, then the framework $(G, \psi, (p, 0^d))$ in $\mathbb{R}^{2d}$ is not $(L, 0^d)$-periodically rigid, where $(L, 0^d) : \Gamma \to \mathbb{R}^{2d}$ maps $\gamma \in \Gamma$ to $(L(\gamma), 0^d)$.

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Observation

For an L-periodically rigid framework (G, ψ, p) in $\mathbb{R}^d$ with rank k periodicity and with $|V(G)| \leq d - k + 1$, $(G, \psi, (p, 0^{D-d}))$ is $(L, 0^{D-d})$-periodically rigid in $\mathbb{R}^D$ for $D \geq d$ since every $(L, 0^{D-d})$-periodic realization of (G, ψ) in $\mathbb{R}^D$ has affine span of dimension at most $|V(G)| + k - 1 \leq d$.

Corollary

Let (G, ψ, p) be a Γ-labeled framework in $\mathbb{R}^d$ with rank k periodicity and $L : \Gamma \to \mathbb{R}^d$. Suppose that (G, ψ, p) is L-periodically rigid and $|V(G)| \leq d - k + 1$. Then (G, ψ, p) is also L-periodically globally rigid.

Theorem (Tanigawa (2016))

For a generically rigid graph $G = (V, E)$, assume that $G - v$ is generically rigid in $\mathbb{R}^d$ and $G - v + K(N_G(v))$ is globally rigid in $\mathbb{R}^d$ for a vertex $v \in V$. Then G is globally rigid in $\mathbb{R}^d$ with $d(v) \geq d + 1$.

Definition

Let (G, ψ) be Γ-labeled and $v \in V$. Assume every edge incident to v is directed from v in $\overrightarrow{E}$.

- $e_1 = vu, e_2 = vw \in \overrightarrow{E} \mapsto e_1 \cdot e_2 = uv$ with label $\psi(vu)^{-1}\psi(vw)$.
- (G_v, ψ_v): Γ-labeled graph obtained from (G, ψ) by removing v and inserting $e_1 \cdot e_2$ for every pair of nonparallel edges e_1, e_2 incident to v.

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Theorem

Let (G, ψ, p) be a generic Γ-labeled framework in $\mathbb{R}^d$ and $L : \Gamma \to \mathbb{R}^d$ be nonsingular. Suppose $v \in V$ has at least $d+1$ neighbors in the covering $(\widetilde{G}, \widetilde{p})$ affinely spanning $\mathbb{R}^d$. Suppose further that

- *$(G - v, \psi|_{G-v}, p|_{V(G)-v})$ is L-periodically rigid in $\mathbb{R}^d$, and*
- *$(G_v, \psi_v, p|_{V(G)-v})$ is L-periodically globally rigid in $\mathbb{R}^d$.*

Then (G, ψ, p) is L-periodically globally rigid in $\mathbb{R}^d$.

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The proof is

- algebraic,
- similar to a proof of a recent lemma by Kaszanitzky, Schulze and Tanigawa.

Theorem (Tanigawa (2016))

Assume that G is 2-rigid in $\mathbb{R}^d$ Then G is globally rigid in $\mathbb{R}^d$.

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A generic L-periodically 2-rigid framework in $\mathbb{R}^d$, is also L-periodically globally rigid in $\mathbb{R}^d$. Moreover, any of its generic L-periodic realizations is L-periodically globally rigid.

Proof.

Induction on $|V|$ using the previous results. □

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(Periodic) body-bar frameworks

- rigid full dimensional bodies connected by disjoint bars
- can be represented by a multigraph H where vertices represent bodies
- corresponding bar-joint framework: take each body as a large complete graph
- rigidity and global rigidity characterized by Tay (1984) and Connelly, Jordán and Whiteley (2013), resp.
- periodic analogue
- represented by a Γ-periodic multigraph $\widetilde{H}$ or Γ-labeled multigraph (H, ψ)
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Theorem

Let $(G_{\widetilde{H}}, \widetilde{p})$ be a generic L-periodic body-bar realization of the multi-graph $\widetilde{H}$ in $\mathbb{R}^d$, and $L : \Gamma \to \mathbb{R}^d$ be nonsingular. Then $(G_{\widetilde{H}}, \widetilde{p})$ is L-periodically globally rigid in $\mathbb{R}^d$ iff $(G_{\widetilde{H}}, \widetilde{p})$ is L-periodically bar-redundantly rigid in $\mathbb{R}^d$.

Proof.

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Thank you for your attention!