Infinitesimal rigidity for unitarily invariant matrix norms

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Bond-node structures:
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- What form does the infinitesimal flex condition take?
- Is infinitesimal rigidity a generic property?

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General norms

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Trace norm

- Maxwell counts, geometric characterisation for $n = 2$

Let M_n denote the vector space of $n \times n$ matrices (over $\mathbb{R}$ or $\mathbb{C}$).

A norm on M_n is unitarily invariant if

$$\|a\| = \|uav\|$$

for all $a \in M_n$ and all unitary matrices $u, v \in M_n$.

Theorem (von Neumann, 1937)

A matrix norm is unitarily invariant if and only if it is obtained by applying a symmetric norm to the vector of singular values of a matrix.

The Schatten p-norms on M_n are defined by,

$$\|a\|_{c_p} = \left(\sum_{i=1}^{n} \sigma_i^p\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|a\|_{c_\infty} = \max_i \sigma_i,$$

where σ_i are the singular values of a.

- c_1 = trace norm
- c_2 = Frobenius norm (= Euclidean norm of matrix entries)
- c_∞ = spectral norm (= operator norm on Euclidean space)

A rigid motion of a normed space $(X, \|\cdot\|)$ is a collection of continuous paths $\alpha = \{\alpha_x : [-1,1] \to X\}_{x \in X}$, with the following properties:

(a) $\alpha_x(0) = x$ for all $x \in X$;

(b) $\alpha_x(t)$ is differentiable at $t = 0$ for all $x \in X$; and

(c) $\|\alpha_x(t) - \alpha_y(t)\| = \|x - y\|$ for all $x, y \in X$ and for all $t \in [-1,1]$.

We write $\mathcal{R}(X, \|\cdot\|)$ for the set of all rigid motions of $(X, \|\cdot\|)$.

Lemma

Let $(X, \|\cdot\|)$ *be a normed space and let* $\alpha \in \mathcal{R}(X, \|\cdot\|)$. *Then,*

(i) *for each* $t \in [-1,1]$ *there exists a real-linear isometry* $A_t : X \to X$ *and a vector* $c(t) \in X$ *such that*

$$\alpha_x(t) = A_t(x) + c(t), \quad \forall\, x \in X.$$

(ii) *the map* $c : [-1,1] \to X$ *is continuous on* $[-1,1]$ *and differentiable at* $t = 0$,

(iii) *for every* $x \in X$, *the map* $A_*(x) : [-1,1] \to X$, $t \mapsto A_t(x)$, *is continuous on* $[-1,1]$ *and differentiable at* $t = 0$, *and,*

(iv) $A_0 = I$ *and* $c(0) = 0$.

Proposition

For any $\alpha \in \mathcal{R}(M_n, \|\cdot\|)$*, there is a neighbourhood* T *of* 0 *in* $[-1,1]$*, and matrices* $u_t, w_t \in U_n$ *and* $c(t) \in M_n$ *for each* $t \in T$*, so that*

(i) $\alpha_x(t) = u_t x w_t + c(t), \quad \forall\, x \in M_n,\ t \in T$;

(ii) $c(0) = 0$ *and* $u_0 = w_0 = I$;

(iii) *the maps* $t \mapsto c(t)$ *and* $t \mapsto u_t x w_t$ *are both differentiable at* $t = 0$*, for any* $x \in M_n$*; and*

(iv) *the maps* $t \mapsto u_t$ *and* $t \mapsto w_t$ *are continuous at* $t = 0$.

A vector field $\eta : X \to X$ of the form $\eta(x) = \alpha'_x(0)$ where $\alpha \in \mathcal{R}(X, \|\cdot\|)$ is referred to as an infinitesimal rigid motion of $(X, \|\cdot\|)$.

Lemma

Let $(X, \|\cdot\|)$ be a normed space and let $\eta \in \mathcal{T}(X, \|\cdot\|)$. Then η is an affine map.

Theorem

If $\eta \in \mathcal{T}(M_n, \|\cdot\|)$, then there exist unique matrices $a, b, c \in M_n$ with $a \in \mathrm{Skew}_n^0$, $b \in \mathrm{Skew}_n$ and $c \in M_n$ so that

$$\eta(x) = ax + xb + c, \quad \forall\, x \in M_n.$$

Define $\Psi : \mathcal{T}(M_n, \|\cdot\|) \to \mathrm{Skew}_n^0 \oplus \mathrm{Skew}_n \oplus M_n$ by setting $\Psi_X(\eta) = (a, b, c)$ if and only if $\eta(x) = ax + xb + c$ for all $x \in X$.

Lemma

Ψ is a linear isomorphism.

Proof.

Let (a, b, c) be in the codomain of Ψ, and for each $x \in M_n$ define

$$\alpha_x : [-1, 1] \to M_n, \quad \alpha_x(t) = e^{ta} x e^{tb} + tc.$$

Since a and b are skew-hermitian, e^{ta} and e^{tb} are unitary for every $t \in \mathbb{R}$, so $\{\alpha_x\}_{x \in \mathcal{M}_n}$ is a rigid motion. The induced infinitesimal rigid motion is the vector field

$$\eta : M_n \to M_n, \quad x \mapsto ax + xb + c.$$

Thus $\Psi(\eta) = (a, b, c)$ and so Ψ is surjective. □

Proposition

$$\dim \mathcal{T}(M_n, \|\cdot\|) = \begin{cases} 2n^2 - n & \textit{if } \mathbb{K} = \mathbb{R}, \\ 4n^2 - 1 & \textit{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

A support functional for a unit vector $x_0 \in X$ is a linear functional $f : X \to \mathbb{R}$ with $\|f\| := \sup\{|f(x)| : x \in X, \|x\| = 1\} \leq 1$, and $f(x_0) = 1$.

Example

Let (G,p) be a bar-joint framework in $(M_n, \|\cdot\|_{c_q})$. Let $vw \in E$, suppose the norm is smooth at $p_v - p_w$ and let $p_0 = \frac{p_v - p_w}{\|p_v - p_w\|_{c_q}}$.

(a) If $q < \infty$, then for all $x \in M_n$,

$$\varphi_{v,w}(x) = \operatorname{trace}(x|p_0|^{q-1}u^*)$$

where $p_0 = u|p_0|$ is the polar decomposition of p_0.

(b) If $q = \infty$, then the largest singular value of the matrix p_0 has multiplicity one. Thus p_0 attains its norm at a unit vector $\zeta \in \mathbb{K}^n$ which is unique (up to scalar multiples). For all $x \in \mathcal{M}_n$, we have

$$\varphi_{v,w}(x) = \langle x\zeta, p_0\zeta \rangle$$

where $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product on $\mathbb{K}^n$.

The norm $\|\cdot\|$ is said to be smooth at $x \in X \setminus \{0\}$ if there exists exactly one support functional at $\frac{x}{\|x\|}$.

Lemma

Let $\|\cdot\|$ be a unitarily invariant norm on M_n, with corresponding symmetric norm $\|\cdot\|_s$ on $\mathbb{R}^n$, and let $x \in M_n$. Then $\|\cdot\|$ is smooth at x if and only if $\|\cdot\|_s$ is smooth at $\sigma(x)$.

A bar-joint framework (G, p) is said to be *well-positioned* in $(X, \|\cdot\|)$ if the norm $\|\cdot\|$ is smooth at $p_v - p_w$ for every edge $vw \in E$.

Proposition

Let (G, p) be a bar-joint framework in $(M_n, \|\cdot\|_{c_q})$.

(i) *If $q \notin \{1, \infty\}$, then (G, p) is well-positioned.*

(ii) *If $q = 1$ then (G, p) is well-positioned if and only if $p_v - p_w$ is invertible for all $vw \in E$.*

(iii) *If $q = \infty$ then (G, p) is well-positioned if and only if $\sigma_1(p_v - p_w) > \sigma_2(p_v - p_w)$ for all $vw \in E$.*

The rigidity map for $G = (V, E)$ and $(X, \|\cdot\|)$ is,

$$f_G : X^V \to \mathbb{R}^E, \quad (x_v)_{v \in V} \mapsto (\|x_v - x_w\|)_{vw \in E}.$$

Lemma

Let (G, p) be a bar-joint framework in a normed linear space $(X, \|\cdot\|)$.

(i) *(G, p) is well-positioned in $(X, \|\cdot\|)$ if and only if the rigidity map f_G is differentiable at p.*

(ii) *If (G, p) is well-positioned in $(X, \|\cdot\|)$ then the differential of the rigidity map is given by*

$$df_G(p) : X^V \to \mathbb{R}^E, \quad (z_v)_{v \in V} \mapsto (\varphi_{v,w}(z_v - z_w))_{vw \in E}.$$

An infinitesimal flex for (G,p) is a vector $u \in X^V$ such that

$$\lim_{t\to 0} \frac{1}{t}\left(f_G(p+tu) - f_G(p)\right) = 0.$$

$\mathcal{F}(G,p)$:= vector space of all infinitesimal flexes of (G,p).

Note that if (G,p) is well-positioned then $\mathcal{F}(G,p) = \ker df_G(p)$.

A non-empty subset $S \subseteq X$ is **full** in $(X, \|\cdot\|)$ if the restriction map

$$\rho_S : \mathcal{T}(X, \|\cdot\|) \to X^S, \quad \eta \mapsto (\eta(x))_{x \in S}$$

is injective.

Lemma

Let $(X, \|\cdot\|)$ be a normed space and let $\emptyset \neq S \subseteq X$. If S has full affine span in X, then S is full in $(X, \|\cdot\|)$.

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We say that a bar-joint framework (G, p) is,

(a) full if $\{p_v : v \in V\}$ is full in $(X, \|\cdot\|)$.

(b) completely full if (G, p), and every subframework (H, p_H) of (G, p) with $|V(H)| \geq 2\dim(X)$, is full in $(X, \|\cdot\|)$.

Given a bar-joint framework (G, p), we define

$$\mathcal{T}(G,p) = \{\zeta\colon V \to X \mid \zeta = \eta \circ p \text{ for some } \eta \in \mathcal{T}(X, \|\cdot\|)\} \subseteq X^V.$$

The elements of $\mathcal{T}(G,p)$ are referred to as the trivial infinitesimal flexes of (G, p).

Lemma

If (G,p) is a full bar-joint framework in $(X, \|\cdot\|)$, then

$$\dim \mathcal{T}(G,p) = \dim \mathcal{T}(X, \|\cdot\|).$$

X	$k(X)$	$l(X)$
$\mathcal{H}_n(\mathbb{R})$	$\frac{1}{2}n(n+1)$	n^2
$\mathcal{M}_n(\mathbb{R})$	n^2	$2n^2 - n$
$\mathcal{H}_n(\mathbb{C})$	n^2	$2n^2 - 1$
$\mathcal{M}_n(\mathbb{C})$	$2n^2$	$4n^2 - 1$

Table: k and l values for admissible matrix spaces.

X	$k(X)$	$l(X)$
$\mathcal{H}_2(\mathbb{R})$	3	4
$\mathcal{M}_2(\mathbb{R})$	4	6
$\mathcal{H}_2(\mathbb{C})$	4	7
$\mathcal{M}_2(\mathbb{C})$	8	15

X	$k(X)$	$l(X)$
$\mathcal{H}_3(\mathbb{R})$	6	9
$\mathcal{M}_3(\mathbb{R})$	9	15
$\mathcal{H}_3(\mathbb{C})$	9	17
$\mathcal{M}_3(\mathbb{C})$	18	35

Table: k and l values for admissible matrix spaces when $n = 2$ and $n = 3$.

A framework (G, p) is infinitesimally rigid if $\mathcal{F}(G, p) = \mathcal{T}(G, p)$.

Theorem

Let (G, p) be a full and well-positioned bar-joint framework in $(M_n, \|\cdot\|)$.

- (i) *If (G, p) is infinitesimally rigid, then $|E| \geq k|V| - l$.*
- (ii) *If (G, p) is minimally infinitesimally rigid, then $|E| = k|V| - l$.*
- (iii) *If (G, p) is minimally infinitesimally rigid and (H, p_H) is a full subframework of (G, p), then $|E(H)| \leq k|V(H)| - l$.*

Theorem

Let (G,p) be a completely full and well-positioned bar-joint framework in $(M_n, \|\cdot\|)$. If (G,p) is minimally infinitesimally rigid then G is (k,l)-tight.

Let $\|\cdot\|$ be a unitarily invariant norm on $X \in \{M_n(\mathbb{K}), H_n(\mathbb{K})\}$ and let $k = \dim X$.

(i) If $\mathbb{K} = \mathbb{R}$, then there exists $p \in X^V$ such that (K_m, p) is full, well-positioned and infinitesimally rigid in $(X, \|\cdot\|)$ for all $m \geq 2k$.

(ii) If $\mathbb{K} = \mathbb{C}$, then there exists $p \in X^V$ such that (K_m, p) is full, well-positioned and infinitesimally rigid in $(X, \|\cdot\|)$ for all $m \geq 2k - 1$.

Thank you