DOUBLE DISTANCE FRAMEWORKS

Stephen Power

Lancaster University

Lancaster, 7th June 2017, Bond-node structures workshop

Joint with Tony Nixon

Flat projective space

$\mathcal{P}_2$: take a flat disc and identify opposite boundary points.
A compact metric space with geodesic distance:

(i) *Euclidean distance*, $d_b(p,q) = \|p-q\|_2$.

(ii) *Re-entrant distance*, $d_r(p,q) = \inf_{|x|=1}\{\|p-x\|_2 + \|q+x\|_2\}$.

(iii) *Geodesic distance*, $d_g(p,q) = \min\{d_b(p,q), d_r(p,q)\}$.

Whence, bar-joint frameworks with two types of bars, for $d_b(\cdot,\cdot), d_r(\cdot,\cdot)$.

The underlying structure graph is 2-coloured: $E = E_b \cup E_r$.

A combinatorial characterisation, à la Laman

Thm. Let (G, p) be a completely regular double-distance framework in $\mathcal{P}_2$ with 2-coloured graph G. The f.a.e.

i) (G, p) is minimally rigid.
ii) G is $(2, 1)$-tight and "limited" (see later).
iii) G has a construction sequence (see later).

Note: $\mathcal{P}_2$ only "has one isometry", rotational, so the (Maxwell) constraints/freedoms count is $|E| = 2|V| - 1$.

Some other double-constraint contexts

The additional constraint $d_2(\cdot, \cdot)$ need not be a metric.

- For $\mathbb{R}^2$: Distance + direction
- For $\mathbb{R}^d$: Euclidean + non-Euclidean distances
- On a surface: geodesic distance + direct distance

Essentially smooth double distance context:
(X, X_0, d_1, d_2) with (X, d_1) a metric space, X_0 a dense smooth manifold, and d_1, d_2 differentiable on $X_0 \times X_0$.

Applied contexts

a) Protein mapping: Residual dipolar coupling (RDC) between rigid units viewed as an additional constraint.

b) 3D sensor networks: Euclidean distances plus altitudes or relative altitudes:

"Toy model": the "separable" double-distance context

$$(\mathbb{R}^3, d_{xy}, d_z)$$

with $d_{xy}(\cdot,\cdot)$ and $d_z(\cdot,\cdot)$ projected distances in the xy-plane and the z-axis.

$(2,3)$, $(2,2)$ and $(2,1)$-tight graphs

1970 Laman/Henneberg

$(2,3)$-tight G: from K_2 by Henneberg moves.

1991 Tay

$(2,2)$-tight G: from K_1 by "Henneberg moves".

$(2,3)$, $(2,2)$ and $(2,1)$-tight graphs

1970 Laman/Henneberg

$(2,3)$-tight G: from K_2 by Henneberg moves.

1991 Tay

$(2,2)$-tight G: from K_1 by "Henneberg moves".

2014 Nixon-Owen-P

$(2,2)$-tight simple G: from K_1 by Henneberg moves, vertex-to-K_4 and vertex-to-4-cycle moves.

$(2,1)$-tight simple G: from $K_5 \backslash e$ by Henneberg, vertex-to-K4, vertex-to-4-cycle, and edge-joining moves.

Proof of the $\mathcal{P}_2$ theorem

A $(2,1)$-tight 2-coloured multi-graph is *limited* if

i) any red subgraph is simple (possibly with looped edges), and
ii) any blue subgraph is $(2,3)$-sparse.

Thm. A limited $(2,1)$-tight multigraph is constructible from a base graph, A_b, A_r or a *loop*, by coloured Henneberg moves and edge joining moves.

Thm. These moves preserve rigidity and A_b, A_r, *loop* are rigid.

0-extensions: OK
1-extensions: Special position arguments for 6 colour cases.

Other directions

Thm. Let (G, p) be a completely regular double-distance framework for $(\mathbb{R}^2, \|\cdot\|_2, \|\cdot\|_q)$, $q \neq 1, 2, \infty$. The f.a.e.

i) (G, p) is minimally rigid.
ii) G is $(2, 2)$-tight and "limited".
iii) G is constructible from K_1 by coloured Henneberg moves !

Further Theory

A) mixed sparsity matroids ?

B) "Protein inspired frameworks" :
body-hinge-pin-bond plus angular constraints.

END