Statistical Analysis of Network Data

James Boyle
supervised by George Bolt

September 4, 2020

Network $G = (V, E)$

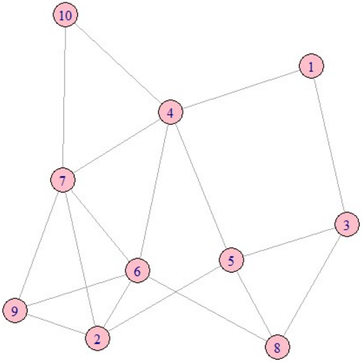

Figure: A graph with $N_V = 10$ vertices

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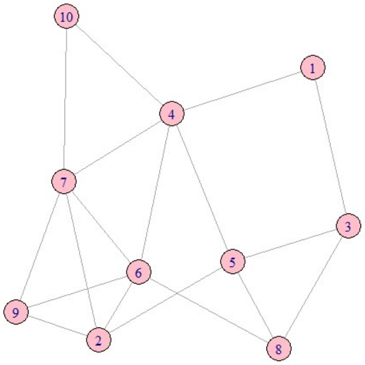
- Vertices $V = \{1, \ldots, N_V\}$



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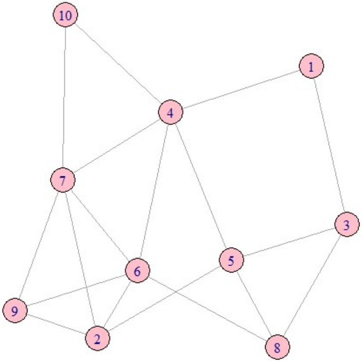
Network Basics

Network $G = (V, E)$

- Vertices $V = \{1, \ldots, N_V\}$
- Edges $\{i, j\}$ joining vertices



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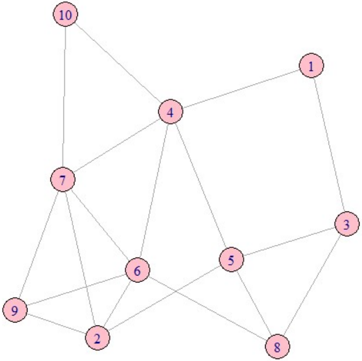

Adjacency matrix $A \in \mathbb{R}^{N_V \times N_V}$

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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- *betweenness centrality* - proportion of shortest paths between pairs of vertices passing through v

Random Graphs

Stochastic Block Model

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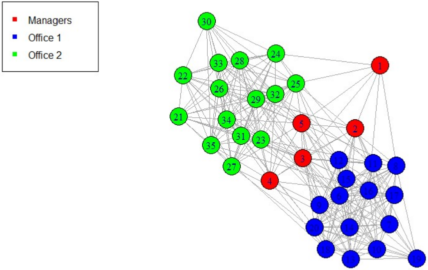
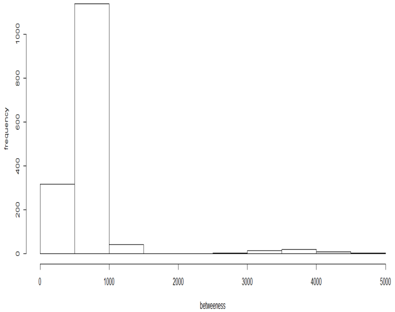
- $\mathbb{P}(\text{edge between vertices } i \text{ and } j) = b_{c_i c_j}$

Stochastic Block Model - Example

- $N_V = 35$
- $K = 3$
- $B = \begin{pmatrix} 0.9 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.05 \\ 0.1 & 0.05 & 0.3 \end{pmatrix}$



Measures of Vertex Centrality



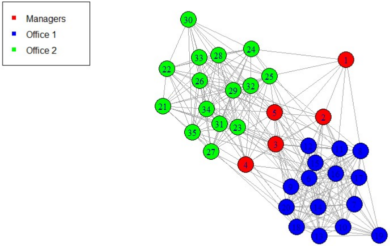


Figure: Betweenness Centrality

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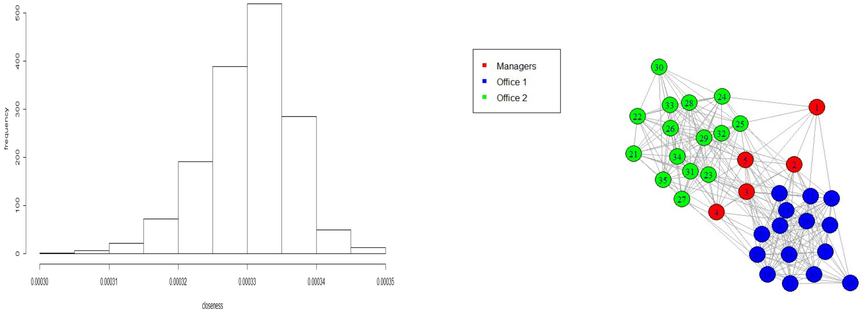

Figure: Closeness Centrality

Multiple Network Models

Single network models are in general not suitable for modelling multiple network observations, e.g. brain scans.

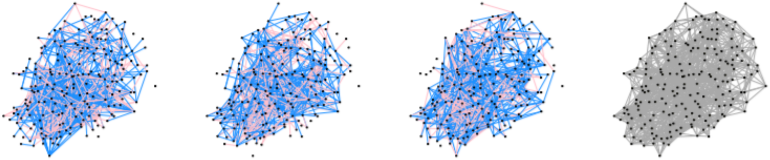

Figure: Brain Networks[4]

Multiple Network Models

Aim: Model multiple noisy realisations of a single "true" network, i.e. observations of the form

$$\text{True Network} + \text{Noise}$$

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For a binary network, noise can only manifest itself in the form of false positive and false negative observations

The Measurement Error Model

Model Assumptions[3][1]:

- True network $A \sim \mathrm{StochasticBlockModel}(N_V, K, B)$

[1] C. M. Le, K. Levin, E. Levina, et al. Estimating a network from multiple noisy realizations.Electronic Journal of Statistics, 12(2):4697–4740, 2018

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Concretely, letting $P, Q \in \mathbb{R}^{K \times K}$ be the matrices of false positive and false negative rates respectively, we suppose that

$$A_{ij}^{(m)} \sim \begin{cases} \mathrm{Bernoulli}(P_{c_i c_j}) & \text{if } A_{ij} = 0 \\ \mathrm{Bernoulli}(1 - Q_{c_i c_j}) & \text{if } A_{ij} = 1 \end{cases}$$

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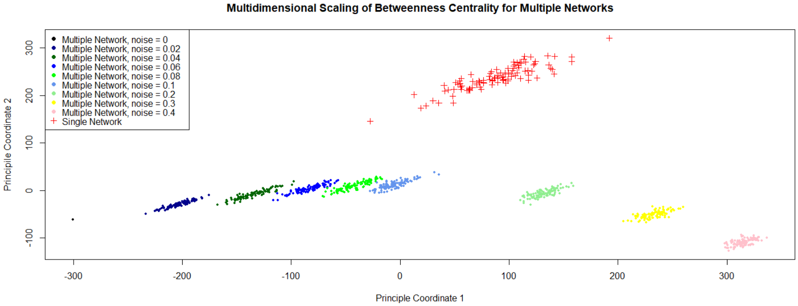

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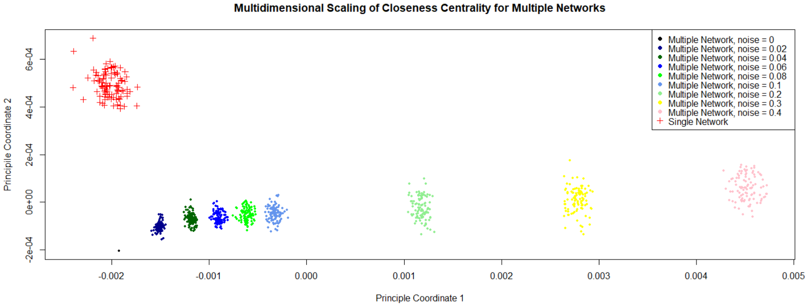

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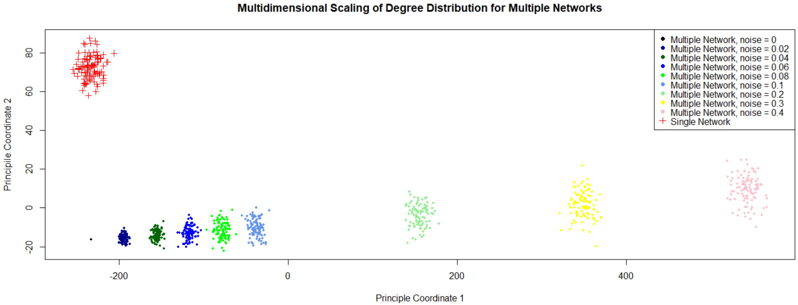

Metric Based Models

Idea[4][2]: Assign probabilities to networks based on their distance from a central, "true", network.

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e.g. For a true network G^{true}, the **Spherical Network Model** assigns

$$\mathbb{P}(G; G^{true}, \gamma) \propto \exp(-\gamma d(G, G^{true}))$$

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On Things not Covered

Network Path data - Each data point is a path through a network

e.g. vertex $\leftrightarrow$ webpage

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Single Network Models

Dynamic networks

Bibliography

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